ODE Math 304-01, Fall 2004

Computer Project #1

Bifurcations With Two Parameters

DUE DATE: Friday, Sept. 24th, in class.

The goal of the project is for you to investigate the qualitative behavior of a family of first-order differential equations involving two parameters. You will make use of certain techniques discussed in class (phase lines, equilibrium points, slope fields and bifurcation diagrams) as well as the mathematical software MAPLE to plot slope fields and approximate solutions to the ODE. Ultimately, you will come up with a parameter plane picture (in the ab-plane) describing the qualitative properties of the family of differential equations

$$\frac{dx}{dt} = ax - x^3 - b$$

where a and b are two independent real parameters.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced at the end of your report. The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Spelling mistakes and sentence fragments, for example, should not occur. Only **one project per group** need be submitted.

Your report should provide coherent answers to each of the following questions. Be sure to attempt to answer all of the questions asked. Read carefully. Please do not overload your report (or my attention for reading) by including large numbers of graphs and tables. A well-written report with a few tables and graphs to illustrate key points is far better than a sloppy report with too many figures.

Useful MAPLE Commands

The MAPLE commands that you will need to use are:

- `plot` to plot ordinary functions of one variable
- `animate` to see animated 2D plots
- `DEplot` to plot slope fields and solutions of an ODE

The basic MAPLE command for plotting graphs of the form $y = f(x)$ is called `plot`. The standard format is

```
plot(function,range,options);
```

where `function` is the function to be plotted — the simplest way to specify one is via a formula, `range` is the range of x-values you want to see plotted, and `options` can be used to control the form of the plot if desired. No options need be specified, however, so that part can be absent. For example,

```
plot(x^4-x^3+2*x-5*sin(x^2),x=1..3,title="My First MAPLE Graph");
```

will plot the function $y = x^4 - x^3 + 2x - 5\sin(x^2)$ on the range $1 \le x \le 3$, and add a title. An example of the `animate` command is given in the Lab Questions below.

The `DEplot` command is part of the `DEtools` package; to use it you will need to start by entering the command `with(DEtools):` to load the `DEtools` routines. Note: the colon at the end of a MAPLE command *suppresses the output* — the command is executed but the results are not displayed. In this case the output would be a (long) list of the commands that are parts of the package.

An Example

Suppose we wanted to study the direction field and solutions of various initial-value problems for the 1st order ODE $y' = ty - 1$. Specifically, say we wanted to plot the solution of the initial-value problem with $y(0) = 1$ for $t \in [-1, 1]$. We could use the following commands: First, define the slope function $f(t, y) = ty - 1$ by typing:

```
f := (t,y) -> t*y - 1;
```

Then, we set up the differential equation in Maple's format:

```
eq := diff(y(t),t) = f(t,y(t));
```

Finally, we use `DEplot` to generate the plot we want:

```
DEplot(eq,y(t),t=-1..1,[[y(0)=1]],linecolor=black);
```

When you execute the `DEplot` command the output will be a plot showing the direction field (with line segments drawn as arrows pointing in the direction of increasing t) together with a plot of the solution.

Remarks:

1. The 3-step process above could actually be combined into a single command if you put the definition of the differential equation directly into the `DEplot` command (instead of building it up using the slope function). I recommend that you do things this way though, at least at first, to keep everything straight in your mind and minimize the chance for typing errors.

2. The general format of the `DEplot` command is

   ```
   DEplot(equation,depvar(indepvar),range,inits,options);
   ```

   where `equation` is the differential equation, in the format given above. After the equation comes the name of the dependent variable (i.e. the unknown function) with the independent variable in parentheses, then the range of values of the independent variable you want to see plotted, then the initial conditions, and options last. *The order of these is fixed – you cannot mix them up freely.* You can plot several different solutions together by putting more than one initial condition inside the outer pair of square brackets, separated by commas, e.g. `[[y(0) = 1],[y(0)=2],[y(-1)=2]]`. Options can be used to control the way solutions are plotted and the appearance of the plot. The `linecolor=black` in the above is an option, for instance. Without that, MAPLE would use a default yellow color that does not show up when you print!

3. To draw the solution, MAPLE is using an approximate numerical method similar to, but more powerful than, the *Euler's Method* you may have studied in calculus. Methods of this type produce a table of values for an approximate solution function by stepping along the slope field. Then MAPLE plots the approximate solution by connecting the points with straight line segments. *This method is far from fool-proof!* In particular, if the slope field is changing rapidly, discontinuous at some points, etc., it can lead to inaccurate results. Fortunately, there are ways to try to fine-tune its behavior. If you notice that your approximate solution looks very jagged (adjacent straight line segments have very different slopes) or if it jumps around wildly, you can try including the option `stepsize=.01` or some number even smaller than .01. The step size is the spacing between successive values of the independent variable when MAPLE computes the approximate solution. Reducing the stepsize from its default value can improve plots. *It also slows down the computation, though, because more points must be computed!* If the slope function is discontinuous at some point on your solution, this may not help; in some cases, this type of method will always fail no matter how small the step size is!

4. *IMPORTANT NOTE:* Some initial conditions with some differential equations lead to solutions that become unbounded extremely quickly. If this happens MAPLE will probably "punt" on trying to draw them and return a blank graph with a message that says: `Floating Point Overflow. Please shorten axes.` This means that you need to reduce the range of values of the independent variable. To learn more about the `DEplot` command, you can type `?DEplot` to get the built-in MAPLE helper.

The Project

All parts of Questions 1-5 in this lab refer to the family of first order ODE's

$$\frac{dx}{dt} = ax - x^3 - b \tag{1}$$

The goal here is to understand the bifurcations that occur in this family and to generate a two-dimensional bifurcation diagram illustrating the possibilities.

1. Set $a = 1$ in the ODE (1).

   a. With $b = -2$, then $b = 0$, then $b = 2$, use the `DEplot` command described above to plot approximate solutions of this equation with initial conditions $x(0) = -2, -1/2, 1/2, 2$ (all on the same axes). Sketch the phase line for this equation with $a = 1$ and each of these b-values (by hand) and explain how they relate to your solution graphs.

   b. Plot $y = x - x^3 - b$ (ordinary function plot) for various b. For another way to see this, you may find it helpful to use animation:

   ```
   with(plots):
   animate(x-x^3-b,x=-2..2,b=-2..2);
   ```

   The toolbar buttons that show up when you left click over the plot region act like the controls on a cassette tape player (remember those??) Use them to play the animation. Note: the default gives 16 frames in the animation — 16 equally spaced b-values starting at $b = -2$, increasing to $b = +2$. Using the information from these plots, sketch the bifurcation diagram for the family $x' = x - x^3 - b$ as b varies (you'll want to do this by hand).

2. Sketch the bifurcation diagram for the family with $a = 0$ and b varying. You may want to repeat calculations as in the previous question to see the pattern. Include any pertinent MAPLE plots you generate in your project report.

3. Sketch the bifurcation diagram for the family with $a = -1$ and b varying. Include any pertinent MAPLE plots you generate in your project report.

4. Now consider the family $x' = ax - x^3 - b$ with $b = -1, 0, 1$ and a varying. Sketch the bifurcation diagram for each of these families. Include any pertinent MAPLE plots you generate in your project report.

5. **Putting it all together** The goal now is to construct a plot of the ab-plane showing the regions where $x' = ax - x^3 - b$ has differing numbers of equilibrium points.

   a. Show that, in general, if $f(x)$ is a polynomial of any degree and $f(x)$ has a multiple root (i.e. double or higher multiplicity) at $x = c$, then

   $$f(c) = f'(c) = 0.$$

   b. Show that for a cubic of the form $f(x) = ax - x^3 - b$, there exists some c with $f(c) = f'(c) = 0$ if and only if

   $$4a^3 - 27b^2 = 0.$$

   The expression $4a^3 - 27b^2$ is called the *discriminant* of this cubic (a generalization of the $b^2 - 4ac$ under the radical sign in the quadratic formula.) A plug: If you take our Abstract Algebra course (MATH 351-352), you may see discriminants of polynomials again in more generality.

   c. What are the different numbers of equilibrium points for our family? Draw a diagram in the ab-plane showing the regions for which there are different numbers of equilibrium points. Be sure to label your diagram, including the type of equilibrium points (eg. sinks, sources or neither).

   d. Using all the information you have acquired thus far, describe the bifurcations that occur when you cross from one region of your diagram into another. Is there a difference depending on whether the cross-over happens at $(0, 0)$ or a point different from $(0, 0)$?

6. Suppose we add on a periodic term as in our discussion of periodic harvesting in class. Consider the differential equation

   $$\frac{dx}{dt} = x - x^3 - 0.1\sin(2\pi t)$$

   Study the solutions and determine whether there appear to be any periodic solutions. Explain.