Mathematical Models MATH 303
Homework Assignment #3
Due Fri., Sept. 28, 5:00 pm

You should write up solutions neatly to all problems, making sure to show all your work. A nonempty subset will be graded. You are encouraged to work on these problems with other classmates, and it is ok to use internet sources for help if it's absolutely necessary (with proper citation); however, the solutions you turn in should be your own work and written in your own words.

Note: Please list the names of any students or faculty who you worked with on the assignment.

1. Read Chapter 3, "Modeling Change One Step at a Time," from the course textbook *Topics in Mathematical Modeling* by K. K. Tung. What is a "chaotic" bank balance and how might it occur?

2. Consider the first-order linear difference equation

$$y_{k+1} = 2y_k - 1, \qquad y_0 = 4\,.$$

   (a) Write out the first 6 terms of the sequence generated by the recursion.

   (b) Using the formula derived in class, find and simplify the solution y_k. (Your answer will only depend on k.) Check your formula works for the first few values of the sequence.

   (c) Suppose the initial condition is changed to $y_0 = 1$. Repeat parts **(a)** and **(b)**. What's different?

3. Consider the first-order linear difference equation

$$y_{k+1} = \frac{1}{3}y_k + \frac{2}{3}, \qquad y_0 = 4\,.$$

   (a) Write out the first 6 terms of the sequence generated by the recursion.

   (b) Using the formula derived in class, find and simplify the solution y_k. Check your formula works for the first few values of the sequence.

   (c) Compute $\lim_{k\to\infty} y_k$.

4. Consider the general first-order linear difference equation $y_{k+1} = ay_k + b$. Let $c = b/(1-a)$.

   (a) Show that if $y_0 = c$, then the resulting sequence repeats itself forever, that is, $y_k = c\ \forall k$. We call $y_0 = c$ a **fixed point** or an **equilibrium**.

   (b) Show that if $|a| < 1$, then for *any* initial condition y_0, $\lim_{k\to\infty} y_k = c$. In this case, c is called an **attracting fixed point** or a **sink**.

5. Working with mortgages.

   (a) Suppose you borrow \$200,000 on a 15-year mortgage at an annual interest rate of 4.25% compounded monthly. What is your monthly mortgage payment? Check that your answer agrees with an online mortgage calculator.

(b) Suppose you borrow \$200,000 on a 30-year mortgage at an annual interest rate of 4.75% compounded monthly. What is your monthly mortgage payment? (Typically, interest rates on a 30-year mortgage are about 0.5% higher than those for a 15-year mortgage.) Check that your answer agrees with an online mortgage calculator.

(c) Find the total amount of interest payed in parts **(a)** and **(b)** (15-year versus 30-year mortgage). How much money do you save by paying off your mortgage in 15 years?

(d) Repeat parts **(a)** and **(b)**, but assume that the interest is compounded continuously.

6. Complete the following exercises from the course textbook:

   Chapter 2 (pp. 65–67): # 2, 5, 6

   Hints: For each problem, try setting up the difference equation first rather than just plugging into formulas. In some cases, you will want to measure time t in months rather than years. Problem #6b is easier than it seems (it is related to #4a).