

Mathematical Models, Spring 2003

Project #1

A Population Model for Whales

DUE DATE: Monday, Feb. 10th, in class.

The goal of the project is for you to use the tools we have developed in the course thus far to analyze the effectiveness of a simple population model for whales. This includes using numerical techniques via a computer as well as performing qualitative and analytical analyses discussed in class. Your overall aim is to assess whether the model would be useful or useless to a population biologist. This may depend on the values of the parameters chosen for the model. It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced at the end of your report.

The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Spelling mistakes and sentence fragments, for example, should not occur. Only one project per group need be submitted.

Your report should provide answers to each of the following questions. Be sure an attempt to answer all of the questions asked. Read carefully. You do not have to include an introduction in your report, although a **conclusion** which states what you learned from the project and further questions you might like to investigate are expected. The bulk of your report should consist of coherent answers to the questions below. Please do not overload your report (or my attention for reading) by including large numbers of graphs and tables. A well-written report with a few tables and graphs to illustrate key points is far better than a sloppy report with too many figures.

The Model

In this project you will explore the following mathematical model used to predict the population of whales:

$$p_{n+1} - p_n = k(M - p_n)(p_n - m)$$

This model is first presented as Exercise #5 in Section 1.2 of the text. Here, p_n represents the number of whales after n years and k, m, M are positive constants with $m < M$. The assumptions for our model are as follows:

- m is the minimal survival rate. If the population falls below m , then the species will become extinct (not enough whales to reproduce the species).
- M is the carrying capacity. If the population is above M , then it should experience a decline (perhaps the food source is not plentiful enough to sustain the population).
- k is a growth rate parameter. If $m < p_n < M$, then the population should grow.

Throughout the project, assume that $m = 100$ and $M = 5000$. You will determine the effectiveness of the model for different values of the growth parameter k .

1. Before doing **any** numerical simulations, perform a qualitative analysis on the model. Does it adequately satisfy the assumptions? What is the sign of δp_n for different values of p_n ? What are the equilibrium populations? What do you expect the stability of the equilibrium solutions to be?
2. For **each** of the following k values, carefully answer the following questions. You should do a numerical analysis (spread sheet, MAPLE, graphs, etc.) as well as give any analytical arguments which help confirm your conclusions. Your justifications for your answers are just as important as your findings. Some items the computer reveals to you can be justified with rigorous mathematical arguments, while other discoveries may be difficult to prove.

k
0.0001
0.0003
0.00045
0.0006

- a. What is the long-term behavior of the population if $p_0 < m$?
- b. What is the long-term behavior of the population if $m < p_0 < M$?
- c. What is the long-term behavior of the population if $p_0 > M$? Be sure to study this region carefully (ie. try some very large populations).
- d. Based on your numerical calculations, classify the equilibrium values as stable or unstable. Do your classifications agree with your answers from question 1? Can you provide any analytical arguments to verify your claims?
- e. Comment on the usefulness of the model for the given k -value. Does the model help you make accurate predictions about the whale population? Is there sensitive dependence on initial conditions? Is the model structurally stable, that is, if the model is perturbed a bit (perhaps k or M changes slightly), do you essentially see the same long-term behavior? Is the Butterfly Effect apparent?