MATH 303, Spring 2003

Homework Assignment #3

DUE DATE: Wednesday, Feb. 12, in class

Homework should be turned in at the BEGINNING OF CLASS. The problems below refer to the supplementary material handed out in class from Chapter 8 of *Differential Equations, 2nd ed.* by Blanchard, Devaney and Hall. You should write up solutions neatly to all problems, making sure to SHOW ALL YOUR WORK. A nonempty subset will be graded. You are strongly encouraged to work on these problems with other classmates, although the solutions you turn in should be your own work. At the top of your homework, please list any students you work with or any other resources you found helpful (websites, books, faculty, etc.)

Systems of Linear Difference Equations

1. Each of the matrices A below represent the coefficient matrix for a difference equation of the form $\mathbf{v_{n+1}} = A\mathbf{v_n}$. Determine the stability of the equilibrium point at the origin (sink, source, saddle, neither).

   a. $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 4.5 & 0 \\ 0 & 0 & 1.47 \end{bmatrix}$

   b. $A = \begin{bmatrix} 0 & 1 \\ 1 & -1.5 \end{bmatrix}$

   c. $A = \begin{bmatrix} 0.25 & -0.25 \\ 3.25 & -1.25 \end{bmatrix}$

2. Consider the car rental company example from the text covered in detail in class (Ex. 1, Section 1.4). Suppose that instead of the percentages given, 30% of the cars rented in Orlando are returned to Orlando and the remaining 70% end up in Tampa. Of the cars rented in Tampa, 50% return to Tampa while the remaining 50% end up in Orlando.

   a. Set up a model for the above dynamical system using a system of linear difference equations. You may use x and y as variables instead of O and T if you desire.

   b. Find the eigenvalues and eigenvectors of the coefficient matrix for your model. Use this information to write down an explicit formula for the solution given any initial condition (x_0, y_0).

   c. How many equilibria does your system have? Where are they? Are they stable, asymptotically stable or unstable? What is the long-term behavior for different initial conditions? (You should be able to solve this analytically but you may wish to use a computer to check your work numerically.)

   d. Does your problem posess any invariance? In other words, are there curves in the xy-plane for which any initial condition starting on that curve will have its orbit remaining on that curve?

e. Finally, compare your answers from above with those of the example from the text and with the solutions generated in class. What are the similarities and differences? You might try graphing some solutions (time-series plots).

3. Notice that in both the previous example and for Ex. 1, Section 1.4 in the text, the coefficient matrix had the property that the sum of each column was one. Explain why this will be true for any percentages chosen for the car rental problem. What assumptions do you have to make about the original problem for this to always be true?

4. The car rental problem is an example of a **Markov Process**. The total number of cars remains fixed and the number of cars in any location never becomes negative. A **Markov matrix** is a matrix with only nonnegative entries and whose columns each sum up to one. The car rental problem will always yield a Markov matrix. For the following, assume that A is a an arbitrary 2×2 Markov matrix.

   a. Show that $\lambda_1 = 1$ is an eigenvalue for A. Conclude that the corresponding Markov process has a line of equilibria.

   b. Show that the remaining eigenvalue λ_2 is real and that $|\lambda_2| \leq 1$. Conclude that the equilibria are stable.

   c. What are the corresponding eigenvectors for λ_1 and λ_2? What information do the eigenvectors provide about the solutions to the corresponding dynamical system $\mathbf{v_{n+1}} = A\mathbf{v_n}$.

Section 8.1 Problems: 1, 3, 6, 7, 10, 15, 19

Section 8.2 Problems: 1, 4, 13, 16, 18, 19, 20, 26

Section 8.4 Problems: 2a, 2b, 2d, 2e, 3, 4, 5