MATH 303 Mathematical Models

Midterm Exam Solutions **Prof. G. Roberts**

1. You have a balance of \$5,000 on a credit card which charges 2% interest per month. You promise to pay p dollars a month to the credit card company and not make any new charges.

   (a) Formulate a model in terms of p which allows you to pay off the credit card in 10 years.
   Ans:
   $$a_{n+1} = 1.02a_n - p, \qquad a_0 = 5000, \quad a_{120} = 0$$
   where a_n represents the balance after n months. Note that 10 years is 120 months, so we want to find p so that $a_{120} = 0$.

   (b) Solve your model analytically to find the value of p (to the nearest cent) which will allow you to pay off the credit card in exactly 10 years.
   Ans: Solving the model above (a linear difference equation) yields
   $$a_n = \frac{-p}{1-1.02} + (1.02)^n \left(5000 - \frac{-p}{1-1.02}\right)$$
   or
   $$a_n = 50p + (1.02)^n(5000 - 50p)$$
   Solving $a_{120} = 0$ for p gives
   $$p = \frac{1.02^{120}}{1.02^{120} - 1} \cdot 100 \approx 110.24$$
   Since we want to be sure the balance is completely paid off, we should make $p = \$110.25$.

   (c) True or False: If p is less than \$100, you will never be able to pay off your credit card. Justify your answer.
   Ans: This is TRUE. One way to think of this is the following: The amount of interest on \$5,000 at 2% is $5,000 \cdot 0.02 = 100$. If the first payment is under \$100, then after one month you won't have paid off even the interest charge, and thus your principle will be BIGGER than \$5,000. Your next interest charge will be even larger than before, and if you continue to pay less than \$100, you will never pay off the credit card. This is why people get so behind in their payments and why credit card companies make so much money.
   A more mathematical answer is: The equilibrium point for the system is $50p$ and the initial condition is $a_0 = 5000$. The equilibrium point is a repellor since $r = 1.02 > 1$. If $a_0 > 50p$, the solution will tend to ∞ (bad for you). Solving $a_0 > 50p$ gives $p < 100$ which means that if you don't contribute more than \$100, the equilibrium point is BELOW the initial condition, causing the solution to head towards ∞.

   (20 pts.)

2. Consider the following predator-prey model for two species x and y. Assume that when the predator and the prey meet, it is good for the population of the predator and bad for the population of the prey.
   $$\begin{aligned} x_{n+1} &= 0.8x_n + 0.001x_ny_n \\ y_{n+1} &= y_n + 0.0001y_n(3000 - y_n) - 0.004x_ny_n \end{aligned}$$

(a) Identify which variable represents the predator and which represents the prey. Explain your answer.

Ans: x is the predator because of the $+0.001x_ny_n$ term in the equation for x_{n+1}. Since this term is positive, interactions between the predator and prey will help x while hurting y because the x_ny_n term has a negative coefficient in the equation for y_{n+1}. Thus, x is the predator (benefits from the species interaction) while y is the prey (hurt by the interaction).

(b) According to the model, what happens to the population of the predator if there are no prey present?

Ans:

If we set $y_n = 0$ in the first equation, we obtain $x_{n+1} = 0.8x_n$ which is the model for exponential decay (since $0.8 < 1$). Therefore, the predators will decay exponentially until becoming extinct.

(c) According to the model, will the population of the prey experience unlimited population growth if there are no predators present?

Ans: The answer is no. If we set $x_n = 0$ in the second equation we obtain

$$y_{n+1} = y_n + 0.0001y_n(3000 - y_n)$$

or

$$\Delta y_n = 0.0001y_n(3000 - y_n)$$

which is the Logistic Population Model with carrying capacity 3,000 and growth rate 0.0001. Assuming that there is no chaotic behavior present, or that a period n-cycle doesn't exist, the long-term behavior of the prey will be to level off at the carrying capacity of 3,000. To be sure that the carrying capacity is indeed an attracting fixed point, we can check the stability using ideas from dynamical systems.

Specifically, let

$$f(y) = y + 0.0001y(3000 - y)$$

be the function for the right-hand side of the logistic equation above. We have that $f(3000) = 3000$ and after a bit of computation $f'(3000) = 0.7 < 1$ so that the fixed point is actually attracting. Thus, no chaos is present and we can be assured that all positive initial conditions for the prey will limit on the carrying capacity of 3,000.

(20 pts.)

3. **(a)** For which value(s) of b will the dynamical system

$$\begin{aligned} w_{n+1} &= -3w_n - z_n \\ z_{n+1} &= 2w_n + bz_n \end{aligned}$$

have a line of equilibrium points?

Ans: Recall that a linear system has a line of equilibria if and only if $\lambda = 1$ is an eigenvalue. Converting the above system into matrix form $\mathbf{v}_{n+1} = Av_n$ gives

$$A = \begin{bmatrix} -3 & -1 \\ 2 & b \end{bmatrix}$$

whose characteristic polynomial is $p(\lambda) = \lambda^2 + (3-b)\lambda + 2 - 3b$. Since we want $\lambda = 1$ to be a root, we set $p(1) = 0$ and solve for b. This gives

$$1 + 3 - b + 2 - 3b = 0$$

or $b = 3/2$.

(b) The orbit of $x_0 = 0$ under iteration of

$$f(x) \;=\; -\frac{3}{2}x^2 + \frac{5}{2}x + 1$$

is periodic. Find the period of this periodic cycle and then determine whether it is attracting, repelling or neutral.

Ans: To find the period of the orbit, just plug in 0 and start iterating. We have $x_1 = f(0) = 1$, then $x_2 = f(1) = 2$ and finally $x_3 = f(2) = 0 = x_0$ so that the period of the orbit is 3. $(0, 1, 2, 0, 1, 2, \ldots)$. To determine if the orbit is attracting or repelling, we must calculate $|f^{3'}(0)|$. BUT, instead of calculating the third iterate of f (an 8th degree polynomial!) and taking the derivative, we use the chain rule. (See class notes.) Since $f'(x) = -3x + 5/2$ we find that

$$|f^{3'}(0)| = |f'(0)| \cdot |f'(1)| \cdot |f'(2)| = \frac{5}{2} \cdot \frac{1}{2} \cdot \frac{7}{2} = \frac{35}{8} > 1$$

so that the orbit is repelling.

(15 pts.)

4. Recall the doubling map $T(x)$ from the homework exercises in Section 8.4,

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x - 1 & \text{if } 1/2 \le x < 1 \end{cases}$$

In your answers to the following questions you may make use of any results proven or discussed concerning the homework.

(a) Find **two** distinct period 3-cycles for $T(x)$.

Ans:

Orbit 1: $1/7, 2/7, 4/7, 1/7, 2/7, 4/7, \ldots$
Orbit 2: $3/7, 6/7, 5/7, 3/7, 6/7, 5/7, \ldots$

There are two ways to obtain this answer: guess or recall that the graph of T^n has lines of slope 2^n. So for example, $T^3(x)$ will have lines of the form $8x, 8x - 1, 8x - 2$ etc. If we solve $8x - 1 = x$ we obtain $x = 1/7$.

(b) Does $T(x)$ possess any attracting periodic cycles? Explain.

Ans:

No. Any point x_0 on an attracting cycle must satisfy $T^n(x_0) = x_0$, that is, be a fixed point for the function T^n. But the slope of the graph of T^n is 2^n. Thus,

$$|T^{n'}(x_0)| = 2^n > 1$$

so that any such point must be repelling.

(c) List the three key properties of a chaotic dynamical system. Explain why a chaotic dynamical system has no attracting periodic cycles.

Ans: A chaotic dynamical system has

(a) A dense set of periodic points.

(b) A dense orbit.

(c) Sensitive dependence on initial conditions.

A chaotic dynamical system can have no attracting cycles because if it did, it would contain a neighborhood U about the cycle which was attracted towards it. Any point in the neighborhood U would not be periodic (violating denseness of periodic points), not exit the neighborhood (violating the existence of a dense orbit), and not iterate towards the periodic cycle (violating sensitive dependence on initial conditions.)

(20 pts.)

5. Consider two skiers Alyssa (A) and Benita (B) skiing down a long smooth hill of constant inclination θ. Assume that the hill is long enough and that both skiers are sufficiently skilled enough so that each will achieve a terminal velocity. Assume that the drag force on a skier is proportional to $Sv^{4/3}$ where S is her surface area and v is her velocity.

(a) Other than air resistance, what are the other forces acting on a skier? Which forces propel the skier down the hill and which forces hold the skier back?

Ans: Gravity propels the skier down the hill as well as a tailwind. Friction between the skiis and the ground will hold the skier back.

(b) Formulate a model relating the weight w of a skier to her terminal velocity v_t. List all the assumptions you make. *Hint:* Recall that terminal velocity is the speed at which your acceleration is zero. Use $F = ma$.

Ans: We will neglect the force due to friction and assume that there is not a tailwind. Friction could be modeled as being proportional to weight and that would not effect the outcome of our model. Assuming geometric similarity between skiers and choosing height h as the characteristic dimension, we have that $S \propto h^2$ and $V \propto h^3$ which implies that

$$S \propto V^{2/3}$$

Assuming constant density, we have that a skiers weight w is proportional to their volume V. Therefore, $S \propto w^{2/3}$.

Now, terminal velocity occurs when acceleration is 0. Using $F = ma$, this means that the total force on the skier must be 0. We want to equate the force due to gravity (proportional to w) with the force due to air resistance (proportional to $Sv^{4/3}$). This yields,

$$w \propto Sv_t^{4/3}$$

where v_t represents the terminal velocity. Substituting in $S \propto w^{2/3}$ gives

$$w^{1/3} \propto v_t^{4/3}$$

or $v_t \propto w^{1/4}$.

(c) If Alyssa weighs 160 lbs and Benita weighs 100 lbs, what is the relationship between their terminal velocities? Does this relationship depend on the slope of the hill? Explain.

Ans: Using our model above, we have

$$\frac{v_{tA}}{v_{tB}} = \left(\frac{w_A}{w_B}\right)^{1/4} = \left(\frac{160}{100}\right)^{1/4} \approx 1.125$$

(25 pts.)