MATH 242: Principles of Analysis

Homework Assignment #9

Partial Solutions

1. Suppose that $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

(a) Show that f is differentiable at $x = 0$ and find $f'(0)$.

(b) Show that f is not differentiable at any other point.

(a) We use the definition of the derivative to compute $f'(0)$. Note that $f(0) = 0$ since $0^2 = 0$. We have

$$f'(0) \;=\; \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x\to 0} \frac{f(x)}{x}.$$

The quantity we are taking the limit of will either be x or 0 depending on whether x is rational or irrational. In either case, the values approach (or equal) 0 as x approaches 0. Thus, we expect that $f'(0) = 0$ and give a rigorous ϵ-δ proof of this fact.

Let $\epsilon > 0$ be given. Set $\delta = \epsilon$. Then,

$$\left| \frac{f(x)}{x} - 0 \right| \;=\; \begin{cases} |x| & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \;<\; \epsilon$$

whenever $0 < |x| < \delta$. This proves that $f'(0) = 0$.

(b) To show that f is not differentiable at any other point, we show that f is not continuous at any point $c \neq 0$. There are two cases. First, if $c \in \mathbb{Q} - \{0\}$, let $x_n = c + \sqrt{2}/n$. This is a sequence of irrationals (since the sum of a rational and an irrational is irrational) that converges to the rational number c. Thus, $f(x_n) = 0$ converges to 0, but $f(c) = c^2 \neq 0$, so $f(x_n)$ does not converge to $f(c)$. This shows that f is not continuous at any $c \in \mathbb{Q} - \{0\}$.

Second, if $c \in \mathbb{I}$, we can construct a sequence of rationals y_n converging to c in the usual fashion. See the solutions for Exercise 4.3.8 on HW #7, for example. Then, $f(y_n) = y_n^2$ converges to c^2 using the BLT for sequences. But $f(c) = 0 \neq c^2$, so again, $f(y_n)$ does not converge to $f(c)$. This shows that f is not continuous at any $c \in \mathbb{I}$.

Putting these two arguments together, we have shown that f is discontinuous at any $c \neq 0$. By the contrapositive to Thm. 5.2.3, this shows that f is not differentiable at any $c \neq 0$.

5.2.2 (a) Let $f(x) = 1/x$. Then, by the definition of the derivative, we have

$$f'(c) \;=\; \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \;=\; \lim_{x\to c} \frac{1/x - 1/c}{x - c} \;=\; \lim_{x\to c} \frac{\frac{c-x}{xc}}{x - c} \;=\; \lim_{x\to c} \frac{-1}{xc} \;=\; \frac{-1}{c^2}.$$

Since c was arbitrary, we have that $f'(x) = -1/x^2$.

(b) To derive the quotient rule, we write $\frac{f(x)}{g(x)}$ as $f(x) \cdot \frac{1}{g(x)}$ and use the product rule to differentiate this last expression. We have

$$\begin{aligned}
\frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right) &= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{-1}{(g(x))^2} \cdot g'(x) \quad \text{using the chain rule and } \mathbf{(a)} \\
&= \frac{f'(x)g(x)}{(g(x))^2} - \frac{g'(x)f(x)}{(g(x))^2} \\
&= \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}
\end{aligned}$$

as desired.

5.2.4 (a) Answer: $a > 0$.

We have, for any value of a, $\lim_{x \to 0^-} f_a(x) = 0$. If $a > 0$, then $\lim_{x \to 0^+} f_a(x) = 0$ agrees with the left-hand limit. Since $f_a(0) = 0$ for this case, we see that

$$\lim_{x \to 0} f_a(x) = f_a(0)$$

and f_a is continuous at $x = 0$.

If $a = 0$, then $\lim_{x \to 0^+} f_a(x) = 1$ which does not equal the left-hand limit. If $a < 0$, then $f_a(0)$ does not exist so the function is not continuous at $x = 0$.

(b) Answer: $a > 1$.

Using the definition of the derivative, assuming that $a > 0$, we have

$$f_a'(0) = \lim_{x \to 0} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x \to 0} \frac{f_a(x)}{x}.$$

As with Exercise #1, we have two cases:

$$\frac{f_a(x)}{x} = \begin{cases} x^{a-1} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Note that this is precisely f_{a-1}. It follows from part **(a)** that $\lim_{x \to 0} \frac{f_a(x)}{x} = 0$ if and only if $a > 1$. In this case, we have $f_a'(0) = 0$. If $a = 1$, then this limit does not exist (different values from the left and right) and if $0 < a < 1$, the right-hand limit approaches ∞ not zero. Away from the point $x = 0$, we can use our usual rules of differentiation to see that (for $a > 1$)

$$f_a'(x) = \begin{cases} ax^{a-1} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

This function is continuous on all of $\mathbb{R}$.

(c) Answer: $a > 2$.

The only tricky point to check is $x = 0$. Using the definition of the derivative, but applied to the function $f_a'(x)$ (for $a > 1$), we have

$$f_a''(0) \;=\; \lim_{x\to 0} \frac{f_a'(x) - f_a'(0)}{x - 0} \;=\; \lim_{x\to 0} \frac{f_a'(x)}{x}.$$

In this case, we have

$$\frac{f_a'(x)}{x} \;=\; \begin{cases} ax^{a-2} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Since the left-hand limit is clearly zero, it follows that $\lim_{x\to 0} \frac{f_a'(x)}{x} = 0$ if and only if $a > 2$. In this case, we have $f_a''(0) = 0$. If $a = 2$, then this limit does not exist (different values from the left and right) and if $1 < a < 2$, the right-hand limit approaches ∞ not zero. Therefore, f_a is twice-differentiable iff $a > 2$.

5.3.4 (a) Proof: Let $h(x) = [f(b)-f(a)]g(x)-[g(b)-g(a)]f(x)$. Since f and g are continuous on $[a,b]$, h, as the difference of continuous functions, is also continuous on $[a,b]$. Since f and g are differentiable on (a,b), h is also differentiable on (a,b) and

$$h'(x) \;=\; [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x)$$

by Theorem 5.2.4 parts (i) and (ii). We compute that

$$h(b) - h(a) \;=\; [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) - [f(b) - f(a)]g(a) + [g(b) - g(a)]f(a) \;=\; 0.$$

In other words, $h(a) = h(b)$. By Rolle's Theorem, there exists a $c \in (a,b)$ such that $h'(c) = 0$. This means that

$$0 \;=\; h'(c) \;=\; [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c)$$

or

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

as desired.

(b) Let $x = g(t)$ and $y = f(t)$ for $a \leq t \leq b$ be parametric equations for a curve in the xy-plane. The endpoints of this curve occur at $t = a$ and $t = b$, and are given by the two points $(g(a), f(a))$ and $(g(b), f(b))$. The slope of the line segment between these two points is $(f(b) - f(a))/(g(b) - g(a))$. From multivariable calculus, the velocity vector along the parametrized curve is given by $g'(t)\,\mathbf{i} + f'(t)\,\mathbf{j}$. The slope of this vector is $f'(t)/g'(t)$. Therefore, the conclusion of the Generalized Mean Value Theorem,

$$\frac{f'(c)}{g'(c)} \;=\; \frac{f(b) - f(a)}{g(b) - g(a)},$$

states that there is some point along the interior of the curve where the slope of the tangent vector equals the slope of the secant line connecting the endpoints of the curve. This is a "generalization" of the Mean Value Theorem from the graph of a one-variable function to any curve in the plane.