MATH 242: Principles of Analysis

Homework Assignment #8

Partial Solutions

2. A *great circle* is a circle on a sphere whose center is the same as the center of the sphere. For example, the equator is a great circle as is any circle passing through both the north and south poles of a sphere. Two points on a sphere are *antipodal* if they are diametrically opposite. For example, the north and south poles are antipodal points. Show that, at any given moment in time, there are two antipodal points with the **same** temperature on any great circle around the Earth. You may assume that the temperature function T is continuous.

Proof: Suppose that $T(x)$ is a continuous function on a circle. We can identify the circle (regardless of its size) with the closed interval $[0, 2\pi]$ in the natural way, assuming that $T(2\pi) = T(0)$. We want to show that there exists a number $c \in [0, \pi]$ such that $T(c + \pi) = T(c)$. We can restrict to $[0, \pi]$ by symmetry. Any solution in the bottom half of the circle $[\pi, 2\pi]$ will have an antipodal point in the top half $[0, \pi]$.

Consider the function $f(x) = T(x+\pi) - T(x)$ on the closed interval $[0, \pi]$. Note that $T(x+\pi)$ is continuous as the composition of continuous functions and consequently, f is continuous as the difference of continuous functions. We have that $f(0) = T(\pi) - T(0)$ and that $f(\pi) = T(2\pi) - T(\pi) = T(0) - T(\pi) = -f(0)$. If $f(0) = 0$, then $T(\pi) = T(0)$ and $c = 0$ is the solution we seek. If $f(0) \neq 0$, then $f(0)$ and $f(\pi)$ have opposite signs. By the Intermediate Value Theorem, there exists a $c \in (0, \pi)$ such that $f(c) = 0$. This implies $T(c + \pi) = T(c)$ as desired.

4.3.2 (a) Proof: Let $\epsilon > 0$ be given. Since g is continuous at $f(c) \in B$, $\exists\, \delta_1 > 0$ such that $|y - f(c)| < \delta_1$ implies $|g(y) - g(f(c))| < \epsilon$. Since f is continuous at c, $\exists\, \delta > 0$ such that $|x-c| < \delta$ implies $|f(x)-f(c)| < \delta_1$ (treating δ_1 as an "epsilon" in the definition of continuity). Putting the two statements together, we have

$$|x - c| < \delta \text{ implies } |f(x) - f(c)| < \delta_1 \text{ implies } |g(f(x)) - g(f(c))| < \epsilon$$

as desired.

(b) Proof: Let $\{x_n\}$ be an arbitrary sequence converging to c with $x_n \in A\ \forall n \in \mathbb{N}$. Since f is continuous at c, the sequence $\{f(x_n)\}$ converges to $f(c)$ (using Thm. 4.3.2, part (iv)). Since g is continuous at $f(c)$ and since $\{f(x_n)\}$ is a sequence in the domain of g converging to $f(c)$, we have that the sequence $\{g(f(x_n))\}$ converges to $g(f(c))$. Since $\{x_n\}$ was arbitrary, we have shown that $g \circ f$ is continuous at c by Thm. 4.3.2, part (iv).

3.3.1 Proof: Suppose that K is a compact set. By the Heine-Borel Theorem, we know that K is both closed and bounded. Since K is bounded above, the $\sup K$ exists by the Axiom of Completeness. This in turn implies that the $\inf K$ exists, using problem #1 on HW #2. Let $s = \sup K$ and let $\epsilon_n = 1/n$. By the Sup Lemma (Lemma 1.3.7), $\exists\, x_n \in K$ such that $s - \epsilon_n < x_n \leq s\ \forall n \in \mathbb{N}$. Using the Squeeze Theorem for Sequences, since both $\{s\}$ and

$\{s - \epsilon_n\}$ converge to s, we have that $x_n \to s$. Thus, by definition, s is a limit point of K. Since K is closed it contains its limit points, and in particular, $s \in K$.

A similar argument works for $u = \inf K$. By our Inf Lemma, $\exists\, y_n \in K$ such that $u \leq y_n \leq u + \epsilon_n\ \forall n \in \mathbb{N}$. Using the Squeeze Theorem for Sequences, since both $\{u\}$ and $\{u + \epsilon_n\}$ converge to u, we have that $y_n \to u$. Thus, by definition, u is a limit point of K. Since K is closed it contains its limit points, and in particular, $u \in K$.

4.4.3 Proof: Suppose that f is a continuous function on a compact set K. Since continuous functions take compact sets to compact sets, $f(K)$ is a compact set. By the previous problem, $s = \sup(f(K))$ and $u = \inf(f(K))$ exist and are contained in the set $f(K)$. This means there exists $x_1 \in K$ with $s = f(x_1)$ and $x_0 \in K$ with $u = f(x_0)$, by the definition of $f(K)$. By the definition of supremum part (i), s is an upper bound for $f(K)$. This means that $s \geq f(x)\ \forall x \in K$. Similarly, by the definition of infimum part (i), u is a lower bound for $f(K)$. This means that $u \leq f(x)\ \forall x \in K$. Putting it all together, we have shown there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)\ \forall x \in K$. This proves the Extreme Value Theorem.

4.4.4 Proof: Suppose that f is continuous on the interval $[a, b]$ with $f(x) > 0\ \forall x \in [a, b]$. By the Extreme Value Theorem, f attains a minimum m for some $c \in [a, b]$. The fact that $f(x) > 0\ \forall x \in [a, b]$ implies that $m > 0$. Therefore, we have that $f(x) \geq m\ \forall x \in [a, b]$. Multiplying both sides of this inequality by the positive quantity $1/(mf(x))$ yields $1/m \geq 1/f(x)\ \forall x \in [a, b]$. Set $M = 1/m > 0$. Since $|1/f(x)| = 1/f(x)$, we have shown that $|1/f(x)| \leq M\ \forall x \in [a, b]$. This shows that $1/f$ is bounded on $[a, b]$.