MATH 242: Principles of Analysis
Homework Assignment #7
Partial Solutions

1. Use the ϵ-δ definition for the limit of a function to prove the following limits:

(a) $\displaystyle\lim_{x\to 3} \frac{2x+3}{7x-3} = \frac{1}{2}$

Preamble: We compute that

$$\left|\frac{2x+3}{7x-3} - \frac{1}{2}\right| \;=\; \left|\frac{-3x+9}{2(7x-3)}\right| \;=\; \frac{3}{2}\cdot\frac{|x-3|}{|7x-3|}\,.$$

We can make the $|x-3|$ term as small as we like by choosing δ appropriately. We need to bound the $1/|7x-3|$ term. To do this, set $\delta = 1$. Then $0 < |x-3| < 1$ implies $2 < x < 4$. On this interval, the function $1/|7x-3|$ is decreasing and has a supremum of $1/11$. Thus we can bound $|f(x) - L|$ by $(3/22)|x-3|$. Choosing $\delta \le (22/3)\epsilon$ will then suffice.

Proof: Let $\epsilon > 0$ be given. Choose $\delta = \min\{1, (22/3)\epsilon\}$. Then,

$$\begin{aligned}
\left|\frac{2x+3}{7x-3} - \frac{1}{2}\right| &= \left|\frac{-3x+9}{2(7x-3)}\right| \\
&= \frac{3}{2}\cdot\frac{|x-3|}{|7x-3|} \\
&< \frac{3}{22}\cdot|x-3| \quad \text{since } \delta \le 1 \\
&< \frac{3}{22}\cdot\frac{22}{3}\epsilon \quad \text{since } \delta \le \tfrac{22}{3}\epsilon \\
&= \epsilon \quad \text{whenever } 0 < |x-3| < \delta\,.
\end{aligned}$$

(b) $\displaystyle\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Proof: Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then, using $|\sin(1/x)| \le 1\ \forall x \in \mathbb{R} - \{0\}$, we have

$$\left|x\sin\left(\frac{1}{x}\right) - 0\right| \;=\; |x|\cdot\left|\sin\left(\frac{1}{x}\right)\right| \;\le\; |x| \;<\; \epsilon \quad \text{whenever } 0 < |x| < \delta\,.$$

4.2.6 Proof: Let $\epsilon > 0$ be given. Since $\displaystyle\lim_{x\to c} g(x) = 0$, there exists a $\delta > 0$ such that

$$|g(x) - 0| \;=\; |g(x)| \;<\; \frac{\epsilon}{M} \quad \text{whenever } 0 < |x-c| < \delta\,.$$

Here we are treating ϵ/M as a particular "epsilon" in the definition of the limit of the function $g(x)$. Then, we have

$$|g(x)f(x) - 0| \;=\; |g(x)| \cdot |f(x)| \;\leq\; M \cdot |g(x)| \;<\; M \cdot \frac{\epsilon}{M} \;=\; \epsilon \quad \text{whenever } 0 < |x - c| < \delta\ .$$

4.2.9 (Squeeze Theorem)

There are two possible proofs, one using the ϵ-δ version of limits, the other using the sequential characterization of limits. The latter seems to be the most straight-forward.

Proof: Let $\{x_n\}$ be an arbitrary sequence contained in A that converges to c such that $x_n \neq c\ \forall n \in \mathbb{N}$. Applying Thm. 4.2.3, we must show that the sequence $\{g(x_n)\}$ converges to L. Since $\lim_{x\to c} f(x) = L$, we know that the sequence $\{f(x_n)\}$ converges to L using Thm. 4.2.3. Similarly, since $\lim_{x\to c} h(x) = L$, we know that the sequence $\{h(x_n)\}$ converges to L, using Thm. 4.2.3 yet again. We are given that $f(x) \leq g(x) \leq h(x)\ \forall x \in A$. Since $\{x_n\} \subseteq A$, we have that $f(x_n) \leq g(x_n) \leq h(x_n)\ \forall n \in \mathbb{N}$. Using the Squeeze Theorem for sequences, it follows that $\{g(x_n)\}$ converges to L, as desired.

4.3.6 Let $g(x)$ be Dirichlet's function. There are two possible arguments to show that g is not continuous at any real number. One is to use the topological characterization of continuity. Another approach is to use the sequential characterization, making two distinct arguments for when $x \in \mathbb{Q}$ and $x \notin \mathbb{Q}$. Both arguments rely on the fact that the rationals and irrationals are each dense in $\mathbb{R}$. We use the topological characterization of continuity, Thm 4.3.2, part (iii).

Proof: Pick $c \in \mathbb{R}$ arbitrarily. Let $\epsilon = 1/2$. Negating property (iii) in Thm. 4.3.2, we must show that $\forall \delta > 0$, $f(V_\delta(c)) \nsubseteq V_\epsilon(g(c))$. In other words, $\forall \delta > 0$, we must find an $x \in V_\delta(c))$ such that $g(x) \notin V_{1/2}(g(c))$. If $c \in \mathbb{Q}$, then since the irrationals are dense in $\mathbb{R}$, for any $\delta > 0$, there exists an irrational $x \in V_\delta(c)$. We have $g(c) = 1$ but $g(x) = 0$. Therefore, $g(x) = 0 \notin V_{1/2}(g(c)) = V_{1/2}(1) = (1/2, 3/2)$.

The argument is similar for $c \notin \mathbb{Q}$. Since the rationals are dense in $\mathbb{R}$, for any $\delta > 0$, there exists a rational $x \in V_\delta(c)$. We then have $g(c) = 0$ but $g(x) = 1$. Therefore, $g(x) = 1 \notin V_{1/2}(g(c)) = V_{1/2}(0) = (-1/2, 1/2)$.

Since c was arbitrary, this shows that the Dirichlet function is nowhere-continuous on $\mathbb{R}$.

4.3.8 (a) Proof: We first show that given any irrational number c, we can construct a sequence of rationals $\{x_n\}$ converging to c. Let $\epsilon_n = 1/n$. Then, for each ϵ_n, there exists a rational x_n satisfying $c - \epsilon_n < x_n < c + \epsilon_n$ since the rationals are dense in $\mathbb{R}$. Since $\epsilon_n \to 0$ as $n \to \infty$, it is straight-forward to show that $x_n \to c$, as desired.

Suppose that f is a function continuous on all of $\mathbb{R}$ with $f(x) = 0$ at any rational number x. Let c be an arbitrary irrational number and let $\{x_n\}$ be a sequence of rationals converging to c. Then, since f is continuous, we have

$$f(c) \;=\; f(\lim_{n\to\infty} x_n) \;=\; \lim_{n\to\infty} f(x_n) \;=\; \lim_{n\to\infty} 0 \;=\; 0.$$

Since c was arbitrary, this shows that $f(x) = 0\ \ \forall x \in \mathbb{R}$.

(b) Yes. **Proof:** Suppose that f and g are continuous on all of $\mathbb{R}$ and $f(x) = g(x)$ for any rational number x. Consider the function $h = f - g$. Then, h is continuous on all of $\mathbb{R}$ since it is the difference of two continuous functions and $h(x) = 0$ for any rational number x. By part **(a)**, this means that $h(x) = 0 \ \ \forall x \in \mathbb{R}$ which in turn implies that $f(x) = g(x) \ \ \forall x \in \mathbb{R}$.