MATH 242: Principles of Analysis

Homework Assignment #6

Partial Solutions

3.2.3 (c) $\{x \in \mathbb{R} : x > 0\}$ is an open set. It is not closed because it does not contain the limit point 0.

(d) The set $A = (0, 1]$ is neither open nor closed. It is not open because any ϵ-neighborhood about 1 will contain elements greater than one and thus not in A. The set is not closed because it does not contain the limit point 0.

(e) The set $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$ is precisely the sequence of partial sums of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. Recall from class that Euler proved this series sums to $\pi^2/6$. It follows that $\pi^2/6$ is a limit point for this set. This set is not open because it contains isolated points. The set is not closed because it does not contain its only limit point $\pi^2/6$, an irrational number.

3.3.2 If a set K is closed and bounded, then K is compact.

Proof: Suppose that K is a closed and bounded set. Let $\{x_n\}$ be an arbitrary sequence in K. We must show that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ that has a limit in K. Since K is bounded, we know that $\{x_n\}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence $\{x_{n_k}\}$ with limit L. We must show $L \in K$. Note that $\{x_{n_k}\}$ is a sequence in its own right and that it is contained in K since it is a subsequence of a sequence in K. There are two cases. If $x_{n_k} = L$ for some k, then $L \in K$ since $x_{n_k} \in K \ \forall k \in \mathbb{N}$. If $x_{n_k} \neq L \ \forall k \in \mathbb{N}$, then by definition, L is a limit point of K. Since K is closed, it contains all of its limit points. In particular, it contains L. Therefore, in either case, $L \in K$. $\square$

3.3.3 The Cantor set is contained inside the closed interval $[0, 1]$ and is thus clearly bounded by 1. It is a closed set because it is formed by the infinite intersection of closed sets (Thm. 3.2.14 (ii)). Alternatively, it is closed since its complement is an open set, the union of open intervals (the removed middle-thirds). By the Heine-Borel Theorem, since the Cantor set is closed and bounded, it is compact.

3.3.7 (a) True. The arbitrary intersection of closed sets is closed by Thm. 3.2.14 (ii). The arbitrary intersection of bounded sets is bounded. To see this, pick an arbitrary element x in the intersection. Since it is in the intersection, it must be in every set. In particular, it is in the first set of the intersection. Since this set is bounded, say by the number M, we have $|x| \leq M$. Since x was arbitrarily chosen, this holds true for any element of the infinite intersection. Therefore, by the Heine-Borel Theorem, the arbitrary intersection of compact sets is compact.

(b) False. Take $A = (0, 1)$ and $K = [-2, 2]$. Then K is compact but $A \cap K = A$ is open and thus not compact.

(c) False. Consider the nested sequence of unbounded closed intervals $F_n = [n, \infty)$. We have $F_{n+1} \subseteq F_n$ $\forall n \in \mathbb{N}$ but the infinite intersection is empty by the Archimedean Property, part (i). If the sets were also bounded, then the result is true (Thm. 3.3.5).

(d) True. A finite set $\{x_1, x_2, \ldots, x_n\}$ will be bounded by $M = \max\{|x_k| : k \in \{1, 2, \ldots, n\}\}$. It is a closed set because it does not have any limit points to contain. Therefore, by the Heine-Borel Theorem, it is a compact set.

(e) False. $\mathbb{Q}$ is a countable set that is not closed nor bounded and thus is not compact. $\mathbb{N}$ is also a countable set that is not compact (not bounded).