MATH 242: Principles of Analysis

Homework Assignment #5

Partial Solutions

1. For each of the infinite series below, determine whether the series converges or diverges. Be sure to state which test you are applying and to verify the hypotheses of the test.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ This series converges using the ratio test. We have that

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+1)!\, n^n}{n!\,(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} \\
&= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} \\
&= \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^{n}\right)^{-1} \\
&= e^{-1} < 1.
\end{aligned}$$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+1}$ This series converges using the Alternating Series Test. Let $a_n = \frac{\sqrt{n}}{n+1}$.

We must show that $\{a_n\}$ is a decreasing sequence that converges to zero.
To show the sequence is decreasing, we must show that

$$\frac{\sqrt{n}}{n+1} \geq \frac{\sqrt{n+1}}{n+2}. \tag{1}$$

Squaring both sides of equation (1) preserves the inequality since both sides are positive for any $n \in \mathbb{N}$. This means that equation (1) is equivalent to

$$\frac{n}{(n+1)^2} \geq \frac{n+1}{(n+2)^2}$$

or

$$\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2} \geq 0.$$

Combining the two fractions, the previous inequality becomes

$$\frac{n(n+2)^2-(n+1)^3}{(n+1)^2(n+2)^2} \;\geq\; 0$$

or

$$\frac{n^2+n-1}{(n+1)^2(n+2)^2} \;\geq\; 0.$$

Since this is clearly valid for $n \geq 1$, we have shown that $\{a_n\}$ is decreasing.
Next, we show that $\lim\limits_{n\to\infty} a_n = 0$ using the BLT for sequences. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} &= \lim_{n\to\infty} \frac{\sqrt{n}/n}{(n+1)/n} \\
&= \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/n} \\
&= \frac{\lim_{n\to\infty} 1/\sqrt{n}}{\lim_{n\to\infty}(1+1/n)} \qquad \text{applying BLT since the limits on top and bottom exist} \\
&= \frac{0}{1+0} \;=\; 0.
\end{aligned}$$

Therefore, by the Alternating Series Test, this series converges. Note that the series is only conditionally convergent, as taking all terms positive gives a series close to one with $a_n = n^{-1/2}$ which diverges by the p-series test.

2. Use the Cauchy Condensation Test to show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

We have $a_n = 1/(n \ln n)$. Note that since n and $\ln n$ are increasing functions of n, a_n is a decreasing sequence of non-negative terms. Applying the Cauchy Condensation Test, we compute

$$2^n a_{2^n} \;=\; \frac{2^n}{2^n \ln 2^n} \;=\; \frac{1}{\ln 2^n} \;=\; \frac{1}{n \cdot \ln 2}.$$

By contradiction, if the infinite series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln 2}$ converged, then by the BLT for series part (i), so would the infinite series $\displaystyle\sum_{n=2}^{\infty} \ln 2 \cdot \frac{1}{n \cdot \ln 2} = \sum_{n=2}^{\infty} \frac{1}{n}$. But this last series is the harmonic series, which diverges (a contradiction.) Therefore, the series obtained by applying the Cauchy Condensation Test diverges and therefore, the original series also diverges.

2.5.3 (c) Using a method similar to how we counted the rationals, consider the sequence:

$$1, 1/2,\; 1, 1/2, 1/3,\; 1, 1/2, 1/3, 1/4,\; 1, 1/2, 1/3, 1/4, 1/5, \ldots\;.$$

This sequence will eventually contain every term of the form $1/n, n \in \mathbb{N}$ and it will contain it infinitely often. Thus, there will be subsequences converging to each element in the set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$, as desired.

(e) This is impossible using the Bolzano-Weierstrass Theorem (BWT). Suppose the sequence $\{x_n\}$ contains a bounded subsequence $\{x_{n_k}\}$. By the definition of a subsequence, $\{x_{n_k}\}$ is itself a sequence with k as the index. Let's rename it $\{y_k\}$. Since $\{y_k\}$ is bounded, it contains a convergent subsequence $\{y_{k_j}\}$ by the BWT. The sequence $\{y_{k_j}\}$ contains terms (chosen in increasing order) from the original sequence $\{x_n\}$ and is thus a convergent subsequence of the original sequence. In other words, the subsequence of a subsequence is also a subsequence of the original sequence. Thus it is not possible to construct a sequence with a bounded subsequence and containing no convergent subsequence.

2.7.4 Let $x_n = 1/n$ and $y_n = 1/n$. Then both infinite series $\sum x_n$ and $\sum y_n$ diverge (harmonic series), yet $\sum x_n y_n = \sum 1/n^2$ is convergent, as proved in class.

Another example is $x_n = 1/n$ and $y_n = 1/\sqrt{n}$.

2.7.8 Suppose that $\displaystyle\sum_{k=1}^{\infty} a_k = A$ and $\displaystyle\sum_{k=1}^{\infty} b_k = B$. Show that $\displaystyle\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof: Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_n = b_1 + b_2 + \cdots + b_n$ be the nth partial sums for the series $\sum a_k$ and $\sum b_k$, respectively. Consider the nth partial sum for the series $\sum(a_k + b_k)$, which is

$$u_n = (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) = (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) = s_n + t_n.$$

Using the BLT for sequences, part (ii), we have

$$\lim_{n\to\infty} u_n \;=\; \lim_{n\to\infty} (s_n + t_n) \;=\; \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n \;=\; A + B,$$

where the last step follows from the definition of convergence of an infinite series. Then, $\displaystyle\lim_{n\to\infty} u_n = A + B$ shows that $\displaystyle\sum_{k=1}^{\infty} (a_k + b_k) \;=\; A + B$, as desired.

2.7.9 (Proving the Ratio Test)

(a) Choose $r' = (r+1)/2$. Then $r < r' < 1$. Let $\epsilon = r' - r > 0$. By the definition of convergence of a sequence, since $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$, there exists an $N \in \mathbb{N}$ such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| \;<\; \epsilon \quad \forall n \geq N.$$

This implies that

$$\frac{|a_{n+1}|}{|a_n|} \;<\; r + \epsilon \;=\; r' \quad \forall n \geq N,$$

which shows that $|a_{n+1}| \leq |a_n| r'$ $\forall n \geq N$. This shows that eventually, the next term in the series is smaller than r' times the previous term, allowing us to compare to a convergent geometric series.

(b) The infinite series $\sum (r')^n$ is a geometric series with ratio $r' < 1$, and thus converges. Multiplying the series by the constant $|a_N|$ does not alter its convergence (BLT for series, part (i)).

(c) We first show by induction that $|a_{N+m}| \leq |a_N|(r')^m$ for any $m \in \mathbb{N}$. The base case is simply $|a_{N+1}| \leq |a_N|r'$ which follows from the fact proven in part (a), choosing $n = N$. For the inductive step, assume that $|a_{N+k}| \leq |a_N|(r')^k$. We must show that $|a_{N+k+1}| \leq |a_N|(r')^{k+1}$. We have

$$|a_{N+k+1}| \;=\; |a_{(N+k)+1}| \;\leq\; |a_{N+k}|r' \;\leq\; |a_N|(r')^k \cdot r' \;=\; |a_N|(r')^{k+1}$$

where the first inequality follows from part (a) and the second from the inductive hypothesis. This proves that $|a_{N+m}| \leq |a_N|(r')^m$ for any $m \in \mathbb{N}$.

Next, we write

$$\sum_{n=1}^{\infty} |a_n| \;=\; \sum_{n=1}^{N} |a_n| \;+\; \sum_{m=1}^{\infty} |a_{N+m}|.$$

The first sum is finite and thus does not effect the convergence of the infinite series. By the fact just proved, the terms of the second series are less than or equal to the terms of the convergent geometric series $\sum |a_N|(r')^m$. By the comparison test, our series converges. $\square$