MATH 242: Principles of Analysis

Homework Assignment #4

Partial Solutions

1. Prove that the sequence $x_n = \frac{1}{8}\sin(\frac{n\pi}{2})$ diverges.

Proof: Suppose by contradiction that the sequence $\{x_n\}$ converges to L. Let $\epsilon = 1/20$. By the definition of convergence, there exists an $N \in \mathbb{N}$ such that $|x_n - L| < 1/20 \ \forall n \geq N$.

First, choose $n \in \{4k-3 : k \in \mathbb{N}\}$ and such that $n \geq N$. For this particular n, we have $x_n = 1/8$. Since $n \geq N$, we also have that $|1/8 - L| < 1/20$. Next, choose $m \in \{2k : k \in \mathbb{N}\}$ and such that $m \geq N$. For this particular m, we have $x_m = 0$. Since $m \geq N$, we also have that $|0 - L| < 1/20$ from the definition of convergence.

Now apply the triangle inequality. We have that

$$1/8 \ = \ |(1/8 - L) + (L - 0)| \ \leq \ |1/8 - L| + |0 - L| \ < \ 1/20 + 1/20 \ = \ 1/10.$$

Since $1/8 \not< 1/10$, we have arrived at a contradiction and the sequence diverges.

2. Prove that if the limit of a sequence exists, then it is unique. (See class notes from 9/24.)

Proof: By contradiction, suppose that $x_n \to L_1$ and $x_n \to L_2$ with $L_1 \neq L_2$. Let $\epsilon = |L_1 - L_2| > 0$. By the definition of convergence applied to $x_n \to L_1$, there exists an $N_1 \in \mathbb{N}$ such that $|x_n - L_1| < \epsilon/3 \ \forall n \geq N_1$. (Note that here we are using $\epsilon/3$ as a particular "epsilon" in the definition of convergence.) Similarly, by the definition of convergence applied to $x_n \to L_2$, there exists an $N_2 \in \mathbb{N}$ such that $|x_n - L_2| < \epsilon/3 \ \forall n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Pick an $n \geq N$. Then both inequalities $|x_n - L_1| < \epsilon/3$ and $|x_n - L_2| < \epsilon/3$ are valid. Using the triangle inequality, we have

$$\epsilon \ = \ |L_1 - L_2| \ \leq \ |L_1 - x_n| + |x_n - L_2| \ < \ \epsilon/3 + \epsilon/3 \ = \ 2\epsilon/3.$$

But since $\epsilon \not< 2\epsilon/3$, we have arrived at a contradiction and $L_1 = L_2$. This shows that the limit of a convergent sequence is unique.

4. Recall that the Fibonacci Series (Sequence) is defined by the recursive relation

$$F_{n+2} = F_{n+1} + F_n, \qquad F_1 = 1, F_2 = 1.$$

Let $G_n = \frac{F_{n+1}}{F_n}$ be the sequence of ratios of consecutive Fibonacci numbers.

a) Prove that $F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1} \ \forall n \in \mathbb{N}$.

b) Prove that $\{G_{2n}\}_{n=1}^{\infty}$ is a decreasing sequence bounded below.

c) Prove that $\{G_{2n-1}\}_{n=1}^{\infty}$ is an increasing sequence bounded above.

d) Prove that the sequence G_n converges and find its limit. Why did I choose G for this sequence? (It is not because that's my first initial!)

a) Proof: Use proof by induction. For the base case, let $n = 1$. We must verify that $F_1F_3 - F_2^2 = (-1)^2$. This is equivalent to $1 \cdot 2 - 1^2 = 1$, which is true. For the induction step, we assume that $F_kF_{k+2} - F_{k+1}^2 = (-1)^{k+1}$ for some $k \in \mathbb{N}$ and show that $F_{k+1}F_{k+3} - F_{k+2}^2 = (-1)^{k+2}$. We have

$$\begin{aligned}
F_{k+1}F_{k+3} - F_{k+2}^2 &= F_{k+1}(F_{k+2} + F_{k+1}) - F_{k+2}^2 \text{ (by definition of the Fibonacci numbers)} \\
&= F_{k+1}^2 + F_{k+1}F_{k+2} - F_{k+2}^2 \\
&= F_{k+1}^2 - F_{k+2}(F_{k+2} - F_{k+1}) \\
&= F_{k+1}^2 - F_{k+2}F_k \text{ (by definition of the Fibonacci numbers)} \\
&= -(F_kF_{k+2} - F_{k+1}^2) \\
&= (-1) \cdot (-1)^{k+1} \text{ (using the inductive hypothesis)} \\
&= (-1)^{k+2} \text{ (rules of exponents).}
\end{aligned}$$

By induction this proves that $F_nF_{n+2} - F_{n+1}^2 = (-1)^{n+1}\ \forall n \in \mathbb{N}$.

Notice that in this proof, we worked from the left-hand side of the equation we wanted to prove and deduced it was equivalent to the right-hand side. This is better form than *assuming* the equation is true to start with and then manipulating it into something else we know is true.

b) Proof: The sequence $\{G_{2n}\}_{n=1}^{\infty} = G_2, G_4, G_6, \ldots$ is the sequence of every other Fibonacci ratio beginning with G_2. We will show $G_{2n} > G_{2(n+1)}\ \forall n \in \mathbb{N}$ directly. This is equivalent to showing $G_{2n} - G_{2n+2} > 0$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
G_{2n} - G_{2n+2} &= \frac{F_{2n+1}}{F_{2n}} - \frac{F_{2n+3}}{F_{2n+2}} \\
&= \frac{F_{2n+1}F_{2n+2} - F_{2n}F_{2n+3}}{F_{2n}F_{2n+2}} \\
&= \frac{F_{2n+1}(F_{2n+1} + F_{2n}) - F_{2n}(F_{2n+2} + F_{2n+1})}{F_{2n}F_{2n+2}} \text{ (by definition of } F_n\text{)} \\
&= \frac{F_{2n+1}^2 - F_{2n}F_{2n+2}}{F_{2n}F_{2n+2}} \\
&= \frac{-(-1)^{2n+1}}{F_{2n}F_{2n+2}} \text{ (using the identity from part } \mathbf{a)}\text{)} \\
&= \frac{1}{F_{2n}F_{2n+2}} \\
&> 0.
\end{aligned}$$

This shows that G_{2n} is decreasing.

To show that $\{G_{2n}\}_{n=1}^{\infty}$ is bounded below, note that $G_n > 0\ \forall n \in \mathbb{N}$ since the Fibonacci numbers are all positive. A better lower bound comes from noticing that

$$G_n \;=\; \frac{F_{n+1}}{F_n} \;=\; \frac{F_n + F_{n-1}}{F_n} \;=\; 1 + \frac{F_{n-1}}{F_n} \;>\; 1 \;\; \forall n \geq 2$$

with the last inequality following from the fact that all the Fibonacci numbers are positive. Since G_n is bounded below so is the subsequence G_{2n}.

c) Proof: The sequence $\{G_{2n-1}\}_{n=1}^{\infty} = G_1, G_3, G_5, \ldots$ is the sequence of every other Fibonacci ratio beginning with G_1. We will show that $G_{2n-1} < G_{2n+1}\ \forall n \in \mathbb{N}$ directly. This is equivalent to showing $G_{2n+1} - G_{2n-1} > 0$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
G_{2n+1} - G_{2n-1} &= \frac{F_{2n+2}}{F_{2n+1}} - \frac{F_{2n}}{F_{2n-1}} \\
&= \frac{F_{2n-1}F_{2n+2} - F_{2n}F_{2n+1}}{F_{2n+1}F_{2n-1}} \\
&= \frac{F_{2n-1}(F_{2n+1} + F_{2n}) - F_{2n}(F_{2n} + F_{2n-1})}{F_{2n+1}F_{2n-1}} \text{ (by definition of } F_n) \\
&= \frac{F_{2n-1}F_{2n+1} - F_{2n}^2}{F_{2n+1}F_{2n-1}} \\
&= \frac{(-1)^{2n}}{F_{2n+1}F_{2n-1}} \text{ (using the identity from part } \mathbf{a)}) \\
&= \frac{1}{F_{2n+1}F_{2n-1}} \\
&> 0.
\end{aligned}$$

This shows that G_{2n-1} is increasing.

To show that $\{G_{2n-1}\}_{n=1}^{\infty}$ is bounded above, we have

$$G_n \;=\; \frac{F_{n+1}}{F_n} \;=\; \frac{F_n + F_{n-1}}{F_n} \;=\; 1 + \frac{F_{n-1}}{F_n} \;\leq\; 2 \;\; \forall n \geq 2,$$

with the last inequality following from the fact that the Fibonacci numbers form an increasing sequence (ie. $F_{n-1}/F_n \leq 1\ \forall n \geq 2$.) Since G_n is bounded above, so is the subsequence G_{2n-1}.

d) Proof: Now the fun part. By the Monotone Convergence Theorem, since both $\{G_{2n}\}$ and $\{G_{2n-1}\}$ are bounded, monotone sequences, they each converge. Suppose that $G_{2n} \to L_{\text{e}}$ and that $G_{2n-1} \to L_{\text{O}}$. We first show that $L_{\text{e}} = L_{\text{O}}$.

We will make use of an important fact. For any $n \in \mathbb{N}$ we have

$$G_{2n} - G_{2n-1} \;=\; \frac{F_{2n+1}}{F_{2n}} - \frac{F_{2n}}{F_{2n-1}} \;=\; \frac{F_{2n-1}F_{2n+1} - F_{2n}^2}{F_{2n}F_{2n-1}} \;=\; \frac{(-1)^{2n}}{F_{2n}F_{2n-1}} \;=\; \frac{1}{F_{2n}F_{2n-1}}.$$

Since F_n is an unbounded and increasing sequence, we can make the denominator of the preceding fraction arbitrarily large and consequently, $G_{2n} - G_{2n-1}$ arbitrarily small.

By contradiction, suppose that $L_{\rm e} \neq L_{\rm O}$ and let $\epsilon = |L_{\rm e} - L_{\rm O}|$. Since $G_{2n} \to L_{\rm e}$, there exists an $N_1 \in \mathbb{N}$ such that $|G_{2n} - L_{\rm e}| < \epsilon/4 \; \forall n \geq N_1$. Likewise, since $G_{2n-1} \to L_{\rm O}$, there exists an $N_2 \in \mathbb{N}$ such that $|G_{2n-1} - L_{\rm O}| < \epsilon/4 \; \forall n \geq N_2$. Finally, by our useful fact above, there exists an $N_3 \in \mathbb{N}$ such that $|G_{2n} - G_{2n-1}| < \epsilon/4 \; \forall n \geq N_3$. Let $N = \max\{N_1, N_2, N_3\}$ and pick any $n \geq N$. Then, by the triangle inequality,

$$\epsilon \;=\; |L_{\rm e} - L_{\rm O}| \;\leq\; |L_{\rm e} - G_{2n}| + |G_{2n} - G_{2n-1}| + |G_{2n-1} - L_{\rm O}| \;<\; \epsilon/4 + \epsilon/4 + \epsilon/4 \;=\; 3\epsilon/4.$$

But this implies $\epsilon < 3\epsilon/4$, a clear contradiction. Therefore, $L_{\rm e} = L_{\rm O}$.

Let $G = L_{\rm e} = L_{\rm O}$. It is straight-forward to show that this is also the limit of the full sequence G_n. Let $\epsilon > 0$ be given. Since $G_{2n} \to G$, there exists an $N_1 \in \mathbb{N}$ such that $|G_{2n} - G| < \epsilon \; \forall n \geq N_1$. Likewise, since $G_{2n-1} \to G$, there exists an $N_2 \in \mathbb{N}$ such that $|G_{2n-1} - G| < \epsilon \; \forall n \geq N_2$. Let $N = \max\{2N_1, 2N_2 - 1\}$. Then, for any even $n \geq N$, we have $|G_n - G| < \epsilon$. But we also have that for any odd $n \geq N$, $|G_n - G| < \epsilon$. Together, this shows that $|G_n - G| < \epsilon \; \forall n \geq N$ and therefore, the sequence $\{G_n\}$ converges.

Finally, to find the value of the special number G we note that

$$G_n \;=\; \frac{F_{n+1}}{F_n} \;=\; \frac{F_n + F_{n-1}}{F_n} \;=\; 1 + \frac{F_{n-1}}{F_n} \;=\; 1 + \frac{1}{G_{n-1}}.$$

Taking the limit of both sides, applying the Big Limit Theorem and using the fact that $\lim_{n\to\infty} G_{n-1} = G$ (shifting back one step doesn't effect the limit), we find that the limit G satisfies the equation

$$G \;=\; 1 + 1/G.$$

Multiplying through by G gives the quadratic equation $G^2 - G - 1 = 0$. This has roots $(1 \pm \sqrt{5})/2$. We throw out the negative root because $G_n > 0 \; \forall n \in \mathbb{N}$. Thus, we have finally proven that the ratio of consecutive Fibonacci numbers converges to the *Golden ratio* $(1 + \sqrt{5})/2$. Cool!

2.3.3 (Squeeze Theorem) Suppose we have three sequences $\{x_n\}, \{y_n\}, \{z_n\}$ satisfying $x_n \leq y_n \leq z_n \; \forall n \in \mathbb{N}$. Suppose further that $x_n \to L$ and $z_n \to L$. Show that y_n converges to L as well.

Note: $|y_n - L| < \epsilon$ is equivalent to $L - \epsilon < y_n < L + \epsilon$. This can be seen by interpreting "$|y_n - L| < \epsilon$" as the distance between the two points y_n and L on the number line must be less than ϵ. Algebraically, it can be derived quickly using the definition of absolute value.

Proof: Let $\epsilon > 0$ be given. Since $z_n \to L$, there exists an $N_1 \in \mathbb{N}$ such that $|z_n - L| < \epsilon \; \forall n \geq N_1$. By the note above, this implies that $z_n < L + \epsilon$ whenever $n \geq N_1$. Since $y_n \leq z_n \; \forall n \in \mathbb{N}$, we have

$$y_n \;<\; L + \epsilon \;\text{ whenever } n \geq N_1. \tag{1}$$

Similarly, since $x_n \to L$, there exists an $N_2 \in \mathbb{N}$ such that $|x_n - L| < \epsilon \; \forall n \geq N_2$. By the note above, this implies that $L - \epsilon < x_n$ whenever $n \geq N_2$. Since $x_n \leq y_n \; \forall n \in \mathbb{N}$, we have

$$L - \epsilon \;<\; y_n \;\text{ whenever } n \geq N_2. \tag{2}$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then both inequalities (1) and (2) are satisfied. Therefore, we have $L - \epsilon < y_n < L + \epsilon$ whenever $n \geq N$ which is equivalent to $|y_n - L| < \epsilon \ \forall n \geq N$. This shows that $y_n \to L$. $\square$

2.3.7 (a) Suppose that $\{a_n\}$ is a bounded (not necessarily convergent) sequence and that $\{b_n\}$ is a convergent sequence with limit $L = 0$. Show that $\{a_n b_n\}$ is a convergent sequence with limit $L = 0$.

Note: We are not allowed to apply the Big (Algebraic) Limit Theorem here because we don't know that $\lim_{n\to\infty} a_n$ exists. For example, if $a_n = n^2$ and $b_n = 1/n$, then $b_n \to 0$ but $a_n b_n = n$ diverges. In order to apply BLT, the limits of the smaller sequences must exist.

Proof: Let $\epsilon > 0$ be given. (Notice that all convergence proofs using the definition start this way!) Since a_n is a bounded sequence, there exists an $M > 0$ such that $|a_n| \leq M \ \forall n \in \mathbb{N}$. Since $b_n \to 0$, there exists an $N \in \mathbb{N}$ such that $|b_n - 0| < \epsilon/M$ whenever $n \geq N$, using the definition of convergence. (Note that here we are using ϵ/M as the "epsilon" in the definition of convergence.)

Punchline:

$$|a_n b_n - 0| \ = \ |a_n b_n| \ = \ |a_n| \cdot |b_n| \ \leq \ M \cdot |b_n| \ < \ M \cdot \epsilon/M \ = \ \epsilon \ \ \forall n \geq N \qquad \square.$$

(b) Suppose that $\{b_n\}$ is a sequence such that $b_n \to b$ with $b \neq 0$. What can we conclude, if anything, about the convergence of the sequence $\{a_n b_n\}$?

Answer: This sequence diverges unless $\lim_{n\to\infty} a_n$ exists. If $a_n \to a$, then $a_n b_n \to ab$ by BLT part (iii). However, suppose $\{a_n\}$ diverges. If $\{a_n b_n\}$ was convergent to some limit L, then by BLT part (iv) we would have

$$\lim_{n\to\infty} a_n \ = \ \lim_{n\to\infty} \frac{a_n b_n}{b_n} \ = \ \frac{\lim\limits_{n\to\infty} a_n b_n}{\lim\limits_{n\to\infty} b_n} \ = \ \frac{L}{b}$$

which implies that $\{a_n\}$ converges. Note that we can apply BLT part (iv) here because $b \neq 0$. This contradicts our assumption. Therefore, $\{a_n b_n\}$ is divergent whenever $\{a_n\}$ is divergent and $b \neq 0$.

2.3.8 (a) Let $a_n = n$ and $b_n = -n$. Each of these diverge but their sum $a_n + b_n = 0$ is the convergent constant sequence of zeroes.

(b) This is impossible. Suppose by contradiction that the sequence $\{x_n + y_n\}$ converges to some limit L. Let x be the limit of the convergent sequence $\{x_n\}$. Then, by the Big Limit Theorem parts (i) and (ii), we have

$$\lim_{n\to\infty} y_n \ = \ \lim_{n\to\infty} (-x_n + (x_n + y_n)) \ = \ -\lim_{n\to\infty} x_n + \lim_{n\to\infty} (x_n + y_n) \ = \ -x + L \ \text{ (exists.)}$$

But this contradicts the fact that $\{y_n\}$ was assumed to be divergent. Therefore, $\{x_n + y_n\}$ must be divergent.

(c) Let $b_n = 1/n$. Then $b_n \neq 0\ \forall n \in \mathbb{N}$ and $\{b_n\}$ is a convergent sequence with limit 0. However, $1/b_n = n$ is a divergent sequence.

(d) This is impossible. Suppose by contradiction that $\{a_n - b_n\}$ is a bounded sequence. This means there exists an $M_1 > 0$ such that $|a_n - b_n| \leq M_1\ \forall n \in \mathbb{N}$. Since $\{b_n\}$ is convergent, it is also bounded by Thm. 2.3.2. Thus, there exists an $M_2 > 0$ such that $|b_n| \leq M_2\ \forall n \in \mathbb{N}$. Using the triangle inequality, we have

$$|a_n| \;=\; |(a_n - b_n) + b_n| \;\leq\; |a_n - b_n| + |b_n| \leq M_1 + M_2 \;\; \forall n \in \mathbb{N}.$$

But this contradicts the fact that $\{a_n\}$ is unbounded. Therefore, $\{a_n - b_n\}$ is an unbounded sequence.

(e) Let $a_n = 1/n$ and $b_n = n$. Then $\{a_n\}$ converges to 0 and $\{b_n\}$ diverges. However, $a_n b_n = 1$ is a convergent sequence.

2.4.2 (a) **Proof:** We first show that the sequence is bounded above by 3 using induction (we will need this to show rigorously that the sequence is decreasing.) Show $x_n \leq 3\ \forall n \in \mathbb{N}$. The base case is satisfied since $x_1 = 3 \leq 3$. Next, assume that $x_k \leq 3$. We must show that $x_{k+1} \leq 3$. Starting with $x_k \leq 3$, multiply both sides by -1 and add 4. This gives $4 - x_k \geq 1$. Dividing both sides of this inequality by the positive number $4 - x_k$ yields $1 \geq 1/(4 - x_k)$ or $x_{k+1} \leq 1 \leq 3$ as desired.

Now we can show that the sequence is decreasing using induction. For the base case, we have to show $x_1 > x_2$. Since $x_1 = 3$ and $x_2 = 1$ this is verified. Next, assume that $x_k > x_{k+1}$. We must show that $x_{k+1} > x_{k+2}$. Starting with the assumption that $x_k > x_{k+1}$, multiply by -1 and add 4 to both sides. This gives $4 - x_k < 4 - x_{k+1}$. Now, using the fact proved above, we know that $x_k \leq 3\ \forall k \in \mathbb{N}$ so the terms on both sides of the inequality are positive. Thus, we can divide both sides by the positive quantity $(4-x_k)(4-x_{k+1})$ to find $1/(4-x_{k+1}) < 1/(4-x_k)$ which is equivalent to $x_{k+1} > x_{k+2}$ as desired.

The sequence is bounded below by zero because $x_n \leq 3\ \forall n \in \mathbb{N}$ implies $4 - x_n \geq 1 > 0$ which implies $x_{n+1} > 0\ \forall n \in \mathbb{N}$. Since $x_1 = 3 > 0$, we have that $x_n > 0\ \forall n \in \mathbb{N}$. Therefore, $\{x_n\}$ is a bounded, monotone sequence and converges to some limit L by the Monotone Convergence Theorem.

(b) **Proof:** We have shown that $x_n \to L$. To see that $\{x_{n+1}\} = x_2, x_3, x_4, \ldots$ also converges to L, let $\epsilon > 0$ be given. By the definition of convergence, since $x_n \to L$, there exists an $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon\ \forall n \geq N$. Using this same choice of N, we have that $|x_{n+1} - L| < \epsilon\ \forall n \geq N$ which verifies the definition of convergence for the sequence $\{x_{n+1}\}$. (The last step follows because if every term after x_N in the sequence $\{x_n\}$ is within ϵ of L, then each term of the sequence $\{x_{n+1}\}$ is within ϵ of L after the $(N-1)$st term.)

All of the above can be summarized nicely by realizing that it is the "tail" of the sequence that we are interested in when studying convergence. Shifting the index up by one just effects the start of the sequence. It does not alter the tail nor the convergence of the sequence.

(c) Taking the limit of both sides of the recursive equation and applying the Big Limit Theorem a few times yields $L = 1/(4 - L)$. Solving this equation leads to the quadratic equation

$L^2 - 4L + 1 = 0$. Using the quadratic formula, we find that $L = (4 \pm \sqrt{12})/2 = 2 \pm \sqrt{3}$. Since the sequence is bounded above by 3, we must have $L \leq 3$, by the Order Limit Theorem (use the constant sequence of 3's). Therefore, $L = 2 - \sqrt{3}$.

2.4.4 The sequence of nested square roots of 2 can be described recursively as $x_{n+1} = \sqrt{2x_n}$ and $x_1 = \sqrt{2}$. We first show that this sequence is bounded above by 2. Specifically, we show that $x_n \leq 2\ \forall n \in \mathbb{N}$. Using induction, the base case $x_1 = \sqrt{2} \leq 2$ is clear. Next, suppose that $x_k \leq 2$. Then $2x_k \leq 4$ and $\sqrt{2x_k} \leq 2$ (we can take the square root of each side since the square root function is increasing.) This last inequality is equivalent to $x_{k+1} \leq 2$ which proves the claim of boundedness.

Next, we show that the sequence is increasing by induction. For the base case, we must show that $x_1 \leq x_2$. This is clear since $2 \leq 2\sqrt{2}$ implies that $x_1 = \sqrt{2} \leq \sqrt{2\sqrt{2}} = x_2$ (again, we can take the square root of each side because the square root function is increasing). Next, assume that $x_k \leq x_{k+1}$. We want to show that $x_{k+1} \leq x_{k+2}$. Beginning with $x_k \leq x_{k+1}$, multiply both sides by 2 and take the square root of each side. This gives $\sqrt{2x_k} \leq \sqrt{2x_{k+1}}$ which is equivalent to $x_{k+1} \leq x_{k+2}$. This shows that the sequence is increasing. By the Monotone Convergence Theorem, since $\{x_n\}$ is increasing and bounded above, it converges to some limit L.

To find the limit, we use the fact that $x_{n+1} \to L$ and apply the Big Limit Theorem to the equation

$$x_{n+1}^2 = 2x_n$$

to avoid the square roots. (Note that we haven't proven that $\lim \sqrt{x_n} = \sqrt{\lim x_n}$ yet.) This gives $L^2 = 2L$ which implies that $L = 0$ or $L = 2$. But since $\{x_n\}$ is increasing and starts with $x_1 = \sqrt{2}$, we can rule out the case $L = 0$. Therefore, the limit of the sequence is 2.