MATH 242: Principles of Analysis

Homework Assignment #3

Partial Solutions

4. The linear function $L(x) = (b-a)x + a$ is a bijection between $(0,1)$ and (a,b). This is clear from a graph. For a more rigorous argument, suppose that $L(x_1) = L(x_2)$. This is equivalent to $(b-a)x_1 + a = (b-a)x_2 + a$. Subtract a and divide by the nonzero quantity $b-a$ to conclude that $x_1 = x_2$. This proves that L is one-to-one. To show that L is onto, pick an arbitrary element $y \in (a,b)$. Let $x = (y-a)/(b-a)$. Since $a < y < b$, we know that $x \in (0,1)$. Then, $L(x) = (b-a)\cdot(y-a)/(b-a) + a = y$. This shows that L is onto and therefore a bijection. We conclude that the open intervals $(0,1)$ and (a,b) have the same cardinality.

5. Show that the open interval $(0,1)$ has the same cardinality as $\mathbb{R}$ by finding a bijection between $(0,1)$ and $\mathbb{R}$. Based on the previous problem, conclude that the entire real number line and any open interval, no matter how small, have the same size!

Proof: Let $f : (0,1) \to (-\pi/2, \pi/2)$ be given by $f(x) = \pi x - \pi/2$ and let $g : (-\pi/2, \pi/2) \to \mathbb{R}$ be given by $g(x) = \tan x$. Then set $h = g \circ f$. Both f and g are bijections. Since the composition of two bijections is a bijection (proved on this HW in problem 1.4.9), we know that h is a bijection from $(0,1)$ to $\mathbb{R}$. This proves that $(0,1) \sim \mathbb{R}$ and using the previous problem as well as problem 1.4.9, we have that $(a,b) \sim \mathbb{R}$ for any open interval (a,b) (no matter how small!).

7. Find the limit of the sequence $x_n = \dfrac{2+5n}{8+11n}$ and prove that it converges to this limit using the ϵ-N defintion of convergence.

The limit is $L = 5/11$. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $N > \frac{18}{121\epsilon}$. Then, if $n \geq N$, we have $n > \frac{18}{121\epsilon}$ which is equivalent to $\frac{18}{121n} < \epsilon$.

Punchline: Therefore,

$$|x_n - L| \;=\; \left|\frac{2+5n}{8+11n} - \frac{5}{11}\right| \;=\; \frac{18}{11(8+11n)} \;<\; \frac{18}{121n} \;<\; \epsilon \;\; \forall n \geq N \quad \square.$$

1.4.9 (a) Given sets A and B, prove that $A \sim B \Rightarrow B \sim A$.

Proof: Since $A \sim B$, there exists a bijection $f : A \to B$. Let $g = f^{-1}$. Since f is one-to-one, f^{-1} exists. We claim that g is a bijection from B to A. Suppose that $g(b_1) = g(b_2)$. This means $f^{-1}(b_1) = f^{-1}(b_2)$. Applying f to both sides and using the fact that $f(f^{-1}(b)) = b \;\; \forall b \in B$ gives $b_1 = b_2$. Thus, g is one-to-one. To show that g is also onto, pick an arbitrary element $a \in A$. We must find a $b \in B$ such that $g(b) = a$. Choose $b = f(a)$. Then $g(b) = f^{-1}(b) = f^{-1}(f(a)) = a$ as desired. This shows that g is a bijection and that $B \sim A$.

(b) Given three sets A, B, C, show that $A \sim B$ and $B \sim C$ implies that $A \sim C$. Taken together with part (a) and the fact that $A \sim A$ (use the identity function), this shows that $\sim$ is an *equivalence relation.*

Proof: It suffices to show that the composition of two bijections is a bijection. For if $A \sim B$, there exists a bijection $f : A \to B$ and if $B \sim C$, there exists a bijection $g : B \to C$, so letting $h = g \circ f$ gives a bijection $h : A \to C$. Suppose that $h = g \circ f$ is the composition of two bijections g and f. First, suppose that $h(a_1) = h(a_2)$. By definition, this means that $g(f(a_1) = g(f(a_2))$. Since g is one-to-one, this means that $f(a_1) = f(a_2)$. Since f is also one-to-one, this implies that $a_1 = a_2$. Thus, h is one-to-one.

To show that h is onto, pick an arbitrary element $c \in C$. Since g is onto, there exists an element $b \in B$ such that $g(b) = c$. But since f is onto as well, there exists an element $a \in A$ such that $f(a) = b$. Thus, we have $h(a) = g(f(a)) = g(b) = c$ as desired. Therefore, h is a bijection and the proof is complete.

2.2.4 The sequence

$$1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, (5 \text{ zeroes})\ , 1, \ldots$$

does not converge to zero.

Proof: Suppose by contradiction that the sequence above converges to $L = 0$. Pick $\epsilon = 1/2$. Then, by the definition of convergence, there exists an $N \in \mathbb{N}$ such that $|x_n - L| = |x_n| = x_n < \epsilon = 1/2\ \forall n \geq N$. However, no matter how large N is, we can always find an $n > N$ with $x_n = 1$. Using this particular n in the previous statement would give $x_n = 1 < 1/2$, a clear contradiction. Therefore, the sequence does not converge to 0. This argument would work for any choice of $\epsilon \leq 1$ since the final step (contradiction) would read $x_n = 1 < \epsilon$ and this is clearly false for any $\epsilon \leq 1$.

Note that for any $\epsilon > 1$, there is always a valid choice of $N \in \mathbb{N}$, namely $N = 1$, that will satisfy the key inequality in the definition of convergence. This is due to the fact that $|x_n - 0| < \epsilon$ becomes either $0 < \epsilon$ or $1 < \epsilon$ (depending on n) and both of these are true if $\epsilon > 1$.

2.2.6 Suppose that for a particular $\epsilon > 0$, we have found a suiable value of $N \in \mathbb{N}$ that "works" for a given sequence in the sense of the definition of convergence.

(a) Then, any *larger* $N \in \mathbb{N}$ will also work in the definition of convergence for the same $\epsilon > 0$. This is because of the $\forall n \geq N$ part of the definition. If something is true for any $n \geq N$ then it also true for any $n \geq N_1 > N$.

(b) Then, this same N will also work for any *larger* value of ϵ. Suppose that $\epsilon_1 > \epsilon$. If $|x_n - L| < \epsilon\ \forall n \geq N$, then it is also true that $|x_n - L| < \epsilon < \epsilon_1\ \forall n \geq N$.