MATH 242: Principles of Analysis

Homework Assignment #2

Partial Solutions

1) Axiom of Completeness (A of C): Any nonempty set of real numbers bounded above has a least upper bound.

Statement L: Any nonempty set of real numbers bounded below has a greatest lower bound.

The point of this problem is to show that the Axiom of Completeness **implies** Statement L. It is also the case that Statement L implies the (A of C) but that is a different problem (exam question?)

First, let's prove a useful fact that we will need throughout the problem: l is a lower bound for A if and only if $u = -l$ is an upper bound for the set $-A$.

To prove this, suppose l is a lower bound for A. This means $l \leq a \ \forall a \in A$. Multiplying both sides of the previous inequality by -1, shows that $-l \geq -a \ \forall a \in A$. But this is the very definition for being an upper bound of the set $-A$. Thus, $u = -l$ is an upper bound for $-A$.

The other direction is similar. Suppose $u = -l$ is an upper bound for $-A$. This means $u \geq -a \ \forall a \in A$. Multiplying both sides of the previous inequality by -1 gives $l \leq a \ \forall a \in A$. But this is the very definition for being a lower bound of the set A. Thus, l is a lower bound for A. This completes the proof of our useful fact.

Proof of a): We know that A is bounded below. Let l be a lower bound for A. Then, by our useful fact, $u = -l$ is an upper bound for $-A$ and $-A$ is bounded above. By the Axiom of Completeness, $-A$ has a least upper bound. In other words, $\sup(-A)$ exists.

Let $s = \sup(-A)$. We must show that $-s = \inf(A)$. To do this, we must show that $-s$ is a lower bound for A and that it is the greatest lower bound. By definition of $\sup(-A)$ part (i), we know that s is an upper bound of $-A$. By our useful fact, this means that $-s$ is a lower bound for A.

Next, let l be any arbitrary lower bound for A. Again, using our useful fact, this means that $-l$ is an upper bound for $-A$. By definition of $\sup(-A)$ part (ii), we have $s \leq -l$ because the supremum is the least of the upper bounds. But $s \leq -l$ implies $-s \geq l$ which shows that $-s$ is greater than any lower bound of A. Therefore, $-s = -\sup(-A)$ satisfies both properties of the definition of infimum for A. Since the infimum of a set is unique, we have $-s = \inf(A)$ as desired.

Proof of b): We want to prove Statement L. Suppose that A is a nonempty set of real numbers bounded below. Consider the set $-A = \{-a : a \in A\}$. By part **a)** and the proof of part **a)**, this set has a supremum s (using A of C) and $-s$ satisfies the definition of $\inf(A)$. Therefore, $\inf(A)$ exists and A has a greatest lower bound.

2) Let A and B be nonempty subsets of $\mathbb{R}$ that are bounded above, and define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Show that $\sup(A + B) = \sup(A) + \sup(B)$.

Proof: First, let $s_a = \sup(A)$ and let $s_b = \sup(B)$. These exist by the Axiom of Completeness. Let $a \in A$ and $b \in B$ be arbitrary. By definition of supremum part (i), we have $a \leq s_a$ and $b \leq s_b$. Then, we have $a + b \leq s_a + b \leq s_a + s_b$. Since a and b were arbitrary, this shows that $s_a + s_b$ is an upper bound for the set $A + B$. By the Axiom of Completeness, this shows that $\sup(A + B)$ exists. Then, by definition of supremum for $A + B$ part (ii), we have $\sup(A + B) \leq s_a + s_b$ since the supremum of a set is always less than or equal to any upper bound of the set.

Next, suppose by contradiction that $\sup(A + B) < s_a + s_b$. Let $\epsilon = s_a + s_b - \sup(A + B)$. By assumption, $\epsilon > 0$. Applying the Sup Lemma to both sets A and B, we know there exists $a \in A$ and $b \in B$, such that $s_a - \epsilon/2 < a$ and $s_b - \epsilon/2 < b$. Adding these two inequalities yields

$$s_a + s_b - \epsilon < a + b.$$

But $s_a + s_b - \epsilon = \sup(A + B)$. Thus, we have found an element $a + b \in A + B$ with $\sup(A+B) < a+b$. This contradicts the first part of the definition of a supremum for $A+B$. Therefore, $\sup(A + B) \geq s_a + s_b$. Taken together with $\sup(A + B) \leq s_a + s_b$, this proves that $\sup(A + B) = s_a + s_b$ as desired.

4) Show that $\displaystyle\bigcap_{n=1}^{\infty} [0, 1/n] = \{0\}$.

Proof: Let $I_n = [0, 1/n]$. Recall that $x \in \displaystyle\bigcap_{n=1}^{\infty} I_n$ only if $x \in I_n \ \forall n \in \mathbb{N}$. Clearly $0 \in I_n \ \forall n \in \mathbb{N}$ since I_n is a closed interval containing 0 as a left endpoint. By contradiction, suppose that $x \in \displaystyle\bigcap_{n=1}^{\infty} I_n$ and $x \neq 0$. Clearly, $x < 0$ is impossible since there are no negative elements in any of the I_n. However, if $x > 0$, there exists an $n \in \mathbb{N}$ such that $1/n < x$ by the Archimedean Property part (ii). For this particular n, we have $x \notin I_n$ and consequently $x \notin \displaystyle\bigcap_{n=1}^{\infty} I_n$. Therefore, $x = 0$ is required and $\displaystyle\bigcap_{n=1}^{\infty} [0, 1/n] = \{0\}$.

1.3.4 Since A and B are nonempty sets that are bounded above, $\sup(A)$ and $\sup(B)$ exist by the Axiom of Completeness. Let $b \in B$ be arbitrary. Since $B \subseteq A$, we have $b \in A$ by definition of subset. Since $\sup(A)$ is an upper bound for A (def. of supremum part (i)), $b \leq \sup(A)$ (def. of upper bound). Since b was arbitrary, this means that $\sup(A)$ is an upper bound for the set B. By definition of supremum for B part (ii), we have $\sup(B) \leq \sup(A)$ because $\sup(B)$ is less than or equal to any upper bound of B. This completes the proof.

1.3.6 **(a)** The set in question is simply $\{1, 2, 3\}$ so $\sup(A) = 3$ and $\inf(A) = 1$.

(b) Choosing $m = 1$ and n arbitrarily large shows that $\sup(B) = 1$. Meanwhile, choosing $n = 1$ and m arbitrarily large shows that $\inf(B) = 0$.

(c) The elements in this set form an increasing sequence $C = \{1/3, 2/5, 3/7, 4/9, \ldots\}$. It is then clear that $\inf(C) = 1/3$ while $\sup(C) = 1/2$.

(d) The smallest and largest values for m and n are 1 and 9 respectively. Thus $\inf(D) = 1/9$ while $\sup(D) = 9$.

1.4.2 **(a)** This follows from the fact that the integers are closed under addition and multiplication and from the rules for adding and multiplying fractions.

(b) By contradiction, suppose that $a + t \in \mathbb{Q}$. Then $\exists r \in \mathbb{Q}$ such that $a + t = r$ or $t = r - a$ or $t = r + (-a)$. By part **(a)**, this means that $t \in \mathbb{Q}$ contradicting the given assumption that $t \in \mathbb{I}$. Thus, $a + t \in \mathbb{I}$. The proof for the product at is similar.

(c) The irrationals are **not** closed under addition and multiplication. Consider $\sqrt{5} + (-\sqrt{5}) = 0$ and $\sqrt{5} \cdot (-\sqrt{5}) = -5$. In each case, two irrationals combine to obtain a rational.

1.4.5 The proof is identical to the one provided for problem **4)** except that 0 is no longer contained in any of the $I_n = (0, 1/n)$ because they are open rather than closed intervals. Thus the infinite intersection of nested **open** intervals can be empty.