MATH 242: Principles of Analysis, Fall 2009

Homework Assignment #1

Partial Solutions

3) We start by showing that $f(C \cup D) \subseteq f(C) \cup f(D)$. Suppose that $y \in f(C \cup D)$. By definition of $f(C \cup D)$, this means that there exists an $x \in C \cup D$ such that $f(x) = y$. By definition of the union of two sets, we have $x \in C \cup D$ implies that $x \in C$ or $x \in D$. This in turn implies that $f(x) \in f(C)$ or $f(x) \in f(D)$, by definition of f applied to a set in the domain. Finally, since $y = f(x)$, we have $y \in f(C)$ or $y \in f(D)$ which by definition of union means $y \in f(C) \cup f(D)$. This shows that $f(C \cup D) \subseteq f(C) \cup f(D)$.

Next we show that $f(C) \cup f(D) \subseteq f(C \cup D)$. Suppose that $y \in f(C) \cup f(D)$. By definition of the union of two sets, this means that $y \in f(C)$ or $y \in f(D)$. By definition of f applied to a set in the domain, this means that either there exists an $x \in C$ such that $f(x) = y$ or there exists an $x \in D$ such that $f(x) = y$. Regardless of which of these is true (it could be both), by definition of union, we have $x \in C \cup D$. This implies that $y = f(x) \in f(C \cup D)$ by definition of f of a set. This shows that $f(C) \cup f(D) \subseteq f(C \cup D)$.

Taken together, $f(C \cup D) \subseteq f(C) \cup f(D)$ and $f(C) \cup f(D) \subseteq f(C \cup D)$ proves that $f(C \cup D) = f(C) \cup f(D)$.

4a) Suppose that $y \in f(C \cap D)$. By definition of f applied to a set, there exists an $x \in C \cap D$ such that $f(x) = y$. By definition of intersection, $x \in C \cap D$ implies $x \in C$ and $x \in D$. By definition of f applied to a set, $f(x) \in f(C)$ and $f(x) \in f(D)$ which means $f(x) \in f(C) \cap f(D)$. Since $y = f(x)$, we have shown $y \in f(C) \cap f(D)$ which proves that $f(C \cap D) \subseteq f(C) \cap f(D)$.

4b) It is interesting to note that the above proof will fail in the reverse direction unless f is a one-to-one function. (Do you see why?) For a counter example showing that the reverse direction does not hold, pick a function that is not one-to-one and choose the sets C and D appropriately. For example, taking $f(x) = \sin x$ and letting $C = [0, 2\pi]$ and $D = [4\pi, 6\pi]$ gives $f(C \cap D) = f(\emptyset) = \emptyset$ while $f(C) \cap f(D) = [-1, 1] \cap [-1, 1] = [-1, 1]$.

5a) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, y^2 \neq x$.

Loosely speaking, the negation of "for all" is "there exists" while the negation of "there exists" is "for all".

Note: The original statement is false since any negative number does not have a square root. Thus, the negation must be true. However, many of you wrote statements that were also false (this a good way to check if your negation is correct.)

5b) $a \in A$ and $\sqrt{a}$ is not an integer.

This is a tricky one. Again, the key thing to remember is that **the negation of a statement has the opposite truth value as the original statement**. Thus, to negate an implication, we need to have it be false in the three instances it is true, and true in the one instant it is false. This means we need to have the hypothesis true **and** the conclusion false. Recall, if the hypothesis is false, the whole statement is automatically true. To negate it, we must have the hypothesis true to begin with.

Another way to obtain the correct negation is to realize that P implies Q is logically equivalent to $\sim P$ or Q. (Check the truth tables of each.) The negation of an "or" statement is an "and" statement. Thus, the negation of "$\sim P$ or Q" is "P and $\sim Q$."

5c) There exists a Holy Cross student who does not eat breakfast at Kimball.

Some of you wrote "No student at Holy Cross eats breakfast at Kimball." This is not the logical negation because it has the same truth value (namely, false) as the original statement. (There are probably students who eat in Hogan and others who eat in Kimball.) Again, to negate a "for all" statement, you use a "there exists" statement.

5d) Bob did not play soccer and John played baseball.

The negation of an "or" statement is an "and" statement (and vice-versa). This is essentially De Morgan's laws. Again, check the truth tables.

6b) The contrapositive: If the Red Sox do not make the playoffs, then either the Yankees or the Tigers will not win their respective divisions.

The negation of an "and" statement is an "or" statement. It would only take one team, the Yankees or the Tigers (not necessarily both), to not win their division to mess up the playoff chances of the Red Sox.

9a) Suppose that $x \in (A \cap B)^c$. By definition of complement of a set, this means $x \notin A \cap B$. This in turn implies that $x \notin A$ or $x \notin B$. We use "or" because it is the negation of the "and" in the definition of intersection. (Alternatively, to not be in the intersection of two sets, you only need to be excluded from one of the two sets.) Then, $x \notin A$ means $x \in A^c$ and $x \notin B$ means $x \in B^c$. Thus, $x \notin A$ or $x \notin B$ can be written as $x \in A^c$ or $x \in B^c$, which, by definition of union of two sets, means $x \in A^c \cup B^c$. This proves $(A \cap B)^c \subseteq A^c \cup B^c$.

9b) The previous proof works perfectly well in the reverse direction. Suppose that $x \in A^c \cup B^c$. By definition of union, this means $x \in A^c$ or $x \in B^c$. By definition of complement of a set, this means $x \notin A$ or $x \notin B$. Now the tricky step. We must have $x \notin A \cap B$ because either $x \notin A$ or $x \notin B$. If your not in a set, then you can't be in anything it is intersected with. Finally, $x \notin A \cap B$ is equivalent to $x \in (A \cap B)^c$. This proves $A^c \cup B^c \subseteq (A \cap B)^c$ which taken with part **a)** shows that $(A \cap B)^c = A^c \cup B^c$.

9c) Note that if we have $C^c = D^c$ for any two sets C and D defined in the same ambient space (eg. the real numbers), then $C = D$ must follow. In other words, if the complements of two sets are equal, then the sets must be equal. This is straight-forward to prove by contradiction. Using the fact that $(A^c)^c = A$ and the result proved in part **b)**, we have

$$(A^c \cap B^c)^c = (A^c)^c \cup (B^c)^c = A \cup B = ((A \cup B)^c)^c$$

Since their complements are equal, we must have $A^c \cap B^c = (A \cup B)^c$ as desired.