MATH 242: Principles of Analysis (Fall 2009)
Practice Problems for the Final Exam

1. Consider the sequence defined recursively by $y_1 = 1$ and $y_{n+1} = 3 - \dfrac{1}{y_n}$ $\forall n \in \mathbb{N}$.

   (a) Use induction to prove that the sequence $\{y_n\}$ is increasing.

   (b) Prove that the sequence $\{y_n\}$ converges. Be sure to state the theroem(s) you are applying.

   (c) Find the limit of the sequence and rigorously prove your claim.

2. Let $A = \{|3 - x| : x \in (-2, 5)\}$. Find $\sup(A)$ and $\inf(A)$.

3. For each of the following sets, state whether the set is finite, countable or uncountable:

$$\mathbb{Q} \cap (0,1), \quad \mathbb{I}, \quad (\frac{1}{101}, \frac{1}{100}), \quad \{\frac{1}{101}, \frac{1}{100}\}, \quad \{\frac{\sqrt{2}}{n^2} : n \in \mathbb{N}\}$$

4. **(a)** Give a sequence of irrational numbers converging to the rational number r.

   (b) Suppose that c is irrational. Prove rigorously that there exists a sequence of rationals converging to c.

   (c) Use the above two results to show rigorously that Dirichlet's function is discontinuous everywhere.

5. Consider the function $f(x)$ defined as $f(x) = \begin{cases} x^2 - x^4 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

   (a) At which points is f continuous? Justify your answer.

   (b) At which points if f differentiable? Justify your answer.

6. You be the Professor. Consider the function

$$f(x) = \begin{cases} 4x & \text{if } x < 3 \\ x^2 - 2x + 3 & \text{if } x \geq 3 \end{cases}$$

   A student walks into your office and gives the following argument that f is differentiable at $x = 3$ and that $f'(3) = 4$. "I've done these problems before. You just differentiate each piece of the function and evaluate at $x = 3$. The first piece of the function is linear and so has derivative 4. The second piece has derivative $2x - 2$ which evaluated at $x = 3$ gives 4, so I get the same answer in each case. Thus, f is differentiable at $x = 3$ and $f'(3) = 4$." Is the student correct? Explain with a rigorous argument.

7. Determine whether the given infinite series converges or diverges using any of the tests from class or the text. Be sure to verify the hypotheses of the test you are applying.

   (a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

   (b) $\sum_{n=1}^{\infty} \cos(\frac{1}{n^2})$

   (c) $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$

8. Determine whether the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ converges absolutely, converges conditionally or diverges.

9. For each of the sets below, decide which of the following attributes apply to the set: open, closed, bounded, compact, or connected

   (a) $A = \{0, 1, 1/2, 1/4, 1/6, 1/8, \ldots\}$

   (b) $B = \mathbb{I} \cap [0, 1]$

   (c) $C = \bigcup_{n=1}^{\infty} [-2 + \frac{1}{n}, 2 - \frac{1}{n}]$

10. Find the following limits and then use the ϵ-δ definition to prove your claim.

    (a) $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$

    (b) $\lim_{x \to -3} \frac{4}{2x+5}$

11. Let f be a twice differentiable function on $[0, 2]$ such that $f(0) = 2$, $f(1) = 0$ and $f(2) = 3$.

    (a) Does there necessarily exist some $c \in (0, 2)$ such that $f(c) = \sqrt{2}$? Justify your answer. What about $f(c) = \pi$?

    (b) Prove that there exists some $c \in (0, 2)$ such that $f''(c) > 0$.

12. Give an example of each of the following, or explain why such an example is impossible. Quote any relevant theorem.

    (a) A function f that is bounded and continuous on $\mathbb{R}$ but attains neither a maximum nor a minimum.

    (b) A function f that is continuous on $[0, 3]$ but attains neither a maximum nor a minimum.

    (c) A function f that is not continuous on $(-1, 1)$, but is continuous at all other points of $\mathbb{R}$.

(d) A sequence that diverges but has an infinite number of subsequences with an infinite number of different limits.

(e) A collection of open sets whose intersection is not open.

(f) A continuous function f that maps $\mathbb{Q}$ onto $[0,1]$.

13. Suppose that f is continuous on $[a,b]$ and $\int_a^b f(x)\,dx = 0$.

(a) Show that there exists at least one point $c \in (a,b)$ such that $f(c) = 0$.

(b) Show that the conclusion of part **(a)** does not necessarily hold if the assumption of continuity is dropped. That is, give an example of a function f such that $\int_a^b f(x)\,dx = 0$, but $f(c) \neq 0\ \forall c \in (a,b)$.

14. Let $h(x) = x^3 - e^{-x}$.

(a) State the Intermediate Value Theorem.

(b) Show that h has a real root.

(c) State the Mean Value Theorem.

(d) Show that there h has **exactly** one root. (*Hint:* Suppose that there were two distinct roots a and b and show this leads to a contradiction.)

15. **(a)** Use induction to prove that

$$\sum_{k=1}^{n} k^2 \;=\; 1^2 + 2^2 + \cdots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}\,.$$

(b) Let $f(x) = x^2$ and let P_n be the partition of $[0,1]$ consisting of n subintervals of equal length. Compute $U(f, P_n)$ and $L(f, P_n)$ using the formula above.

(c) Without resorting to the Fundamental Theorem of Calculus, use part **(b)** to prove that f is integrable on $[0,1]$ and find the value of the integral.

16. True or False: If true give a proof (or reference the appropriate theorem), if false give a counterexample or correct the statement.

(a) The $\sqrt{p}$ is irrational for any prime number p.

(b) The sequence $\{\cos(e^n)\}$ is divergent, but contains a convergent subsequence.

(c) If A and B are any two sets of real numbers, then $(A \cup B)^C = A^C \cup B^C$.

d) All Cauchy sequences are bounded.

(e) If $f(x)$ and $g(x)$ are positive functions over the interval $[a,b]$, then

$$\int_a^b f(x)\cdot g(x)\,dx \;=\; \int_a^b f(x)\,dx \;\cdot\; \int_a^b g(x)\,dx$$

f) If a function $g(x)$ is differentiable, then its derivative is continuous.

g) If f is continuous on $\mathbb{R}$ and $f(r) = 3$ for any rational number r, then $f(x) = 3\ \forall x \in \mathbb{R}$.

h) $e > \pi$