MATH 242: Principles of Analysis (Fall 2009)

Practice Problems for Exam 1

1. Prove that $\sqrt[3]{2}$ is irrational.

2. Use induction to prove that $n^2 \leq 4^n$ for all $n \geq 1$.

3. Let P_n be the statement that $2 + 6 + 10 + \cdots + (4n - 2) = 2(n^2 + 1)$.

   a) Show that if P_k is true, then P_{k+1} is true.

   b) Is P_n true $\forall n \in \mathbb{N}$?

4. Form the logical negation of the following statements:

   a) If the Red Sox when the pennant, then Yankees fans will be sad.

   b) There is a student who does not like Principles of Analysis.

   c) $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|x_n - L| < \epsilon \; \forall n \geq N$.

5. True or False (if true prove it, if false find a counterexample):

   For any two sets A and B, $B = A - (A - B)$.

6. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. Statement L: Every nonempty set of real numbers bounded below has a greatest lower bound. Prove that Statement L implies the Axiom of Completeness (see HW #2, problem #1.)

8. Given two sets A and B of $\mathbb{R}$, define the set $A \ominus B$ to be

   $$A \ominus B = \{a - b : a \in A \text{ and } b \in B\}.$$

   True or False (if true prove it, if false find a counterexample): $\sup(A \ominus B) = \sup(A) - \sup(B)$.

9. Let $A = \bigcap_{n=1}^{\infty} (1, 3 + \frac{1}{n})$. Find $\sup(A)$ and prove your assertion.

10. Prove that the composition of two bijections is a bijection.

11. Use the definition of convergence to prove that the following sequences converge.

    a) $x_n = \dfrac{9 - 7n}{8 + 13n}$.

    b) $x_n = \dfrac{\sin(n^2)}{3^n}$.

12. Suppose that $\lim_{n \to \infty} x_n = 5$ and $\lim_{n \to \infty} y_n = 7$. Prove that there exists an $N \in \mathbb{N}$ such that

    $x_n + y_n < 13 \; \forall n \geq N$.

13. Suppose that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y.

a) Use the ϵ-N definition of convergence to prove that $\{x_n - y_n\}$ converges to $x - y$.

b) Use the Big Limit Theorem to prove that $\{x_n - y_n\}$ converges to $x - y$.

14. Suppose that $x_n \geq 0 \ \forall n \in \mathbb{N}$. Prove that if $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}$. Note that there are two cases here: $L = 0$ and $L \neq 0$. Try multiplying by the conjugate in the case $L \neq 0$.

15. Consider the sequence defined recursively by $x_1 = 1$ and

$$x_{n+1} = \frac{x_n^2 + 8}{6}.$$

a) Prove that $\{x_n\}$ is an increasing sequence.

b) Prove that $x_n < 3 \ \forall n \in \mathbb{N}$

c) Prove that $\{x_n\}$ converges and find its limit, being careful to specify what theorems you are using.

16. True or False: If true give a proof, if false give a counterexample or correct the statement.

a) If the sequences $\{x_n\}$ and $\{y_n\}$ each diverge, then so does $\{x_n y_n\}$.

b) The irrationals are closed under addition (eg. the sum of any two irrationals is irrational).

c) The set of irrationals in any open interval are uncountable.

d) $\lim_{n\to\infty} \frac{\sqrt{2n^2+1}}{\sqrt[3]{3n^3+1}} = \frac{\sqrt{2}}{\sqrt[3]{3}}$.

e) All convergent sequences are also monotone.

f) All monotone sequences are convergent.