MATH 242 Principles of Analysis

Exam #1 SOLUTIONS October 8, 2009 Prof. G. Roberts

1. **(a)** Give the definition of the *infimum* of a set A. (6 pts.)

   Answer: x is the *infimum* of A if

   (i) x is a lower bound for A ($x \le a \ \ \forall a \in A$)

   (ii) $x \ge b$ for any lower bound b of A.

   (b) Let $B = \bigcap_{n=1}^{\infty} (2 - \frac{1}{n}, 5)$. Find $\inf(B)$. (No proof required.) (5 pts.)

   Answer: $\inf(B) = 2$. Since $2 \in (2 - \frac{1}{n}, 5)\ \forall n \in \mathbb{N}$, we have that $2 \in B$. Any number r less than 2 will not be in B by the Archimedean Property. It follows that $B = [2, 5)$ and $\inf(B) = 2$.

   (c) Let $C = \{x \in \mathbb{Q} : x^3 < 2\}$. Find $\sup(C)$. (No proof required.) (5 pts.)

   Answer: $\sup(C) = \sqrt[3]{2}$. The set C is bounded above by 2 since $2^3 > 2$. By the Axiom of Completeness, C has a supremum. Solving $x^3 = 2$ for x gives the supremum of the set. Recall that a supremum does not have to be in the set so it is fine that $\sqrt[3]{2}$ is irrational.

2. **(a)** Carefully form the negation of the following statement: (6 pts.)

   $\forall y > 0, \exists x < 0$ such that $y = x^2$.

   Answer: The negation is $\exists y > 0$ such that $\forall x < 0,\ y \neq x^2$. Note that the original statement is true and our negation is false (choose $x = -\sqrt{y}$).

   (b) Suppose $B = (2, 3]$ and $f(x) = |x - 1|$. Find $f^{-1}(B)$. (6 pts.)

   Answer: $[-2, -1) \cup (3, 4]$. Recall that $|a - b|$ measures the distance between a and b. To find $f^{-1}(B)$, we seek all points that are within 2 and 3 units of the point 1. 2 units away from 1 are the points 3 and -1 while 3 units away from 1 are the points 4 and -2. This gives the boundaries of $f^{-1}(B)$ and from there, the solution follows quickly.

3. Find the $\lim_{n\to\infty} x_n$ for the sequence

$$x_n = \frac{\sin(2n)}{n^3} + 2$$

and prove that $\{x_n\}$ converges to your limit using the ϵ-N definition of convergence. (15 pts.)

Answer: The limit is 2 using BLT part (ii). We then calculate that

$$|x_n - 2| = \left|\frac{\sin(2n)}{n^3} + 2 - 2\right| = \left|\frac{\sin(2n)}{n^3}\right| \le \frac{1}{n^3}.$$

We want the last term to be less than ϵ. Solving $1/n^3 < \epsilon$ for n gives $n > \sqrt[3]{1/\epsilon}$.

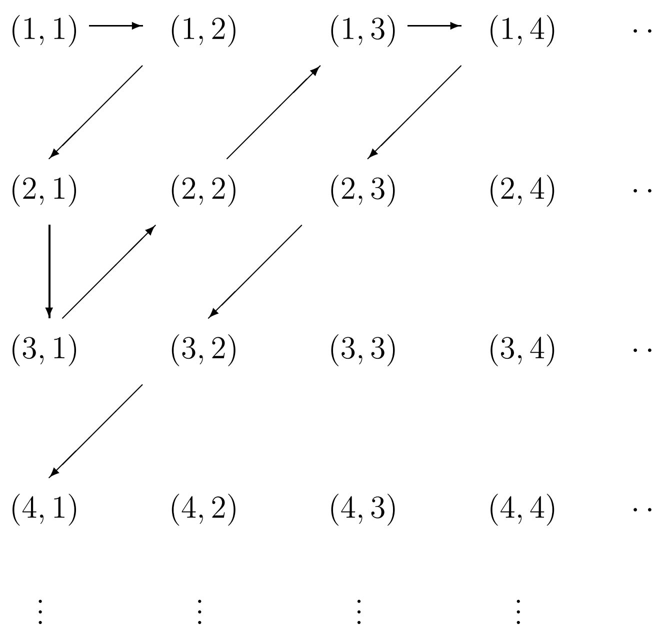
Proof of Convergence: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $N > \sqrt[3]{1/\epsilon}$. Then, if $n \ge N, n > \sqrt[3]{1/\epsilon}$ which is equivalent to $1/n^3 < \epsilon$.

Punchline:

$$|x_n - 2| = \left|\frac{\sin(2n)}{n^3} + 2 - 2\right| = \left|\frac{\sin(2n)}{n^3}\right| \le \frac{1}{n^3} < \epsilon \quad \forall n \ge N \qquad \square.$$

4. Consider the set $\mathbb{N}^2 = \mathbb{N}\times\mathbb{N} = \{(m,n) : m,n \in \mathbb{N}\}$. (This is called the Cartesian product.) For example, the points $(2,7)$ and $(56,2009)$ (ordered pairs **not** open intervals) are each elements of $\mathbb{N}\times\mathbb{N}$ but the points $(-3,1/2)$ and $(2,\sqrt{2})$ are not. Is $\mathbb{N}\times\mathbb{N}$ countable or uncountable? Justify your claim. (12 pts.)

   Answer: The set $\mathbb{N}\times\mathbb{N}$ is countable. It can be counted in a similar manner to how we counted the rationals, using Pete's zig-zag method to ensure we count all the elements in $\mathbb{N}\times\mathbb{N}$.



   More precisely, let $B_k = \{(m,n) : m,n \in \mathbb{N} \text{ and } m+n=k\}$. For each k, B_k is a finite set. Then, we have $\mathbb{N}\times\mathbb{N} = \bigcup_{k=2}^{\infty} B_k$ since any element $(m,n)\in\mathbb{N}\times\mathbb{N}$ will be contained in the set B_{m+n}. To actually list the elements in $\mathbb{N}\times\mathbb{N}$, we start by listing all the elements in B_2, then B_3, then B_4, etc. In this manner, we construct a bijection from $\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$.

5. Consider the sequence $x_1 = 0.27, x_2 = 0.2727, x_3 = 0.272727, \ldots, x_n = 0.27\cdots27$ (27 repeated n times). Note that this sequence satisfies the recursive relation $100x_{n+1} - 27 = x_n \;\forall n \in \mathbb{N}$. You may assume that this sequence is bounded above by 1.

   (a) Prove that this sequence is increasing. (7 pts.)

   Answer: Note that $100x_{n+1} - 27 = x_n$ is equivalent to $x_{n+1} = \frac{x_n+27}{100}$.

   Claim: $x_n \le x_{n+1} \;\forall n \in \mathbb{N}$.

   Proof by Induction: First, we have $x_1 = 0.27 \le 0.2727 = x_2$ so the base case is satisfied. Next, we assume that $x_k \le x_{k+1}$ and show that $x_{k+1} \le x_{k+2}$. Starting with $x_k \le x_{k+1}$, add 27 to both sides to obtain $x_k + 27 \le x_{k+1} + 27$. Then divide both sides of the previous inequality by 100. This gives $\frac{x_k+27}{100} \le \frac{x_{k+1}+27}{100}$ which is equivalent to $x_{k+1} \le x_{k+2}$ $\square$.

(b) Prove that this sequence converges and find the limit. Be sure to state precisely which theorems you are applying. (8 pts.)

Answer: Since $\{x_n\}$ is an increasing sequence bounded above by 1, it converges by the Monotone Convergence Theorem. Let the limit of this sequence be L. Note that $\lim_{n\to\infty} x_{n+1} = L$ as proven on homework. Taking the limit as $n \to \infty$ of the equation $100x_{n+1} - 27 = x_n$ and applying the Big Limit Theorem leads to $100L - 27 = L$. The solution to this linear equation is $L = 3/11$. Therefore, $\lim_{n\to\infty} x_n = 3/11$. One can check on a calculator that $3/11 = 0.27272727\cdots$, as expected.

6. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, provide a **counterexample** or justification for your choice. (10 pts. each)

(a) The number $\sqrt{5}$ is irrational.

Answer: True. Using proof by contradiction, suppose that $\sqrt{5}$ was rational. Then we can write $\sqrt{5} = p/q$ for some integers p and q. This in turn implies that $5q^2 = p^2$. The prime factorization of p^2 will have an even number of 5's (or none) because the square of a natural number will contain twice as many primes in its factorization as the original number. But the prime factorization of $5q^2$ will have an odd number of 5's because of the extra 5 in front. This violates the Fundamental Theorem of Arithmetic which states that the prime factorization of a natural number is *unique* up to reordering. Thus, $5q^2 = p^2$ is impossible and $\sqrt{5}$ is irrational.

(b) The number of rationals in the interval $(4, 4.01)$ is finite.

Answer: False. The rationals are dense in the real numbers. By contradiction, if the number of rationals in the interval $(4, 4.01)$ were finite, then we could find a sub-interval containing no rationals at all. This contradicts the fact that for any $a, b \in \mathbb{R}$ with $a < b$, there exists an $r \in \mathbb{Q}$ such that $a < r < b$.

Another way to see that there are an infinite number of rationals in the interval $(4, 4.01)$ is to consider the sequence $x_n = 4 + \frac{1}{n^2}$ for $n \geq 11$. Each term in the sequence is rational since the rationals are closed under addition. Since $4 < x_n < 4.01 \ \forall n \geq 11$, we have an infinite number of rationals in $(4, 4.01)$.

(c) Suppose that $\lim_{n\to\infty} x_n = 5$ and that $\lim_{n\to\infty} (3x_n - y_n) = 9$. Then, it follows that $\lim_{n\to\infty} y_n = 6$.

Answer: True. This one is a little tricky. You cannot apply the BLT to expand $\lim_{n\to\infty} (3x_n - y_n)$ because we don't know that $\lim_{n\to\infty} y_n$ exists. Instead, we apply BLT to a special multiple of the two sequences we know converge. Specifically, we have

$$\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} (3x_n - (3x_n - y_n)) \\
&= 3 \lim_{n\to\infty} x_n - \lim_{n\to\infty} (3x_n - y_n) \qquad \text{(using BLT parts (i) and (ii))} \\
&= 3 \cdot 5 \ - \ 9 \\
&= 6 \quad \Box.
\end{aligned}$$