MATH 242: Principles of Analysis
Theorems on Infinite Series

Def. 2.4.3: Convergence for an Infinite Series

The infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ converges. In this case, we say that

$$\sum_{n=1}^{\infty} a_n = S \quad \text{if and only if} \quad \lim_{n\to\infty} s_n = S.$$

One metaphor I like for infinite series (thanks to Prof. Levandosky) is the infinite shopping cart going through check-out at a supermarket. Each item you purchase is a term a_n. As the scanner records the price of each item, the subtotal (eg. sum of all items purchased thus far) is displayed on the screen for you and the cashier to see. The list of numbers being displayed on the screen is precisely the sequence of partial sums. Each item you purchase increases your total ($a_n > 0$ and $s_n > s_{n-1}$) while if you have any coupons or store credit, this will reduce your total ($a_n < 0$ and $s_n < s_{n-1}$). As you watch your infinite number of purchases go by, the question is whether the sequence of sub-totals converges or not. If it diverges, you either get in an argument with the cashier over the total price (because it never settles down to a particular value) or you go broke (because the sum is infinite).

Most theorems about infinite series can be deduced by applying the correct theorem about converging sequences to the sequence of partial sums. In particular, the Cauchy Criterion (CC) is particularly useful when proving some of the theorems below. This is because it is the *tail* of the infinite series that matters when trying to determine convergence. Even if the first 2 million terms in the series are extremely large, the series can still converge if the remaining terms approach zero fast enough.

Def. 2.7.8: Absolute and Conditional Convergence

An infinite series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges. On the other hand, the series **converges conditionally** if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges.

Note that $a_n \leq |a_n|$ always, so that considering the series of absolute values is considering a series with larger terms. Having a mixture of positive and negative terms in a series (eg. alternating $+-+-+-\ldots$) is *useful* for convergence as it helps the sequence of partial sums converge. Note also that if a series has only positive terms, or a finite number of negative terms, then absolute convergence is equivalent to regular convergence.

The following are all examples of absolutely convergent series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad \sum_{n=1}^{\infty} e^{-n}.$$

The standard example of a conditionally convergent series is the **Alternating Harmonic Series**, defined as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \;=\; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - \cdots$$

This series converges by the Alternating Series Test (see below) and its sum is exactly $\ln 2$. However, taking absolute values of the terms gives the Harmonic Series which diverges (as proved in class). Another example of a conditionally convergent series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}.$$

1. **Big Limit Theorem for Series** (Thm. 2.7.1)

   If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

   (i) $\sum_{n=1}^{\infty} c a_n = cA \quad \forall c \in \mathbb{R}$

   (ii) $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

   Note that we do not have a theorem about the product or quotient of two infinite series (as we do with the BLT for sequences) as this is a far more delicate matter. See Section 2.8 if you would like to learn more about the product of two infinite series.

2. **Cauchy Criterion for Series** (Thm. 2.7.2)

   The infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

   $$|a_{m+1} + a_{m+2} + \cdots + a_n| \;<\; \epsilon$$

   for all natural numbers m, n satisfying $n > m \geq N$.

   This test is just a restatement of the CC for sequences. It gets to the heart of the matter. No matter how small we take ϵ, we can go out far enough in the series such that the sum of the remaining terms (the infinite tail) is less than ϵ.

3. ***n*th Term Test** (Thm. 2.7.3)

   If $\lim_{n\to\infty} a_n \neq 0$, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges. Equivalently, if the infinite series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

This is really a test for *divergence*. If the terms we continue to add to the series are not approaching zero, then the partial sums will continue to grow (or perhaps oscillate), and the series will diverge. From the shopping cart perspective, if we keep adding items whose price is not getting close to zero, then our total will either grow forever or perhaps oscillate continually.

Very Important: The converse of the n-th term test is false! Just because the terms in an infinite series go to zero, does *not* mean the series converges. The counterexample is the all-important **Harmonic Series** which has terms converging to zero but still sums to infinity. Having the terms in an infinite series head to zero is *necessary* for convergence but not *sufficient*. You need to apply some other test to check for convergence.

4. **Comparison Test** (Thm. 2.7.4)

   Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences satisfying $0 \leq a_n \leq b_n \ \forall n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. Equivalently, if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

   This is pretty clear. Given two infinite series of positive terms, if the one with bigger terms converges, so does the smaller one. The contrapositive is that if the smaller one diverges, than so must the bigger one. It is worth pointing out that this theorem is still valid if the sequences obey $0 \leq a_n \leq b_n \ \forall n \geq N$ for some natural number N, rather than $\forall n \in \mathbb{N}$. As long as the two series eventually obey the inequality, then the conclusion holds. Remember, it is the *tail* of the infinite series that matters not a finite number of terms at the start.

5. **Geometric Series** (Example 2.7.5)

   A geometric series with ratio r and first term a,

$$\sum_{n=1}^{\infty} ar^{n-1} \ = \ a + ar + ar^2 + ar^3 + \cdots ,$$

   converges if and only if $|r| < 1$. In the case of convergence, the sum is $\frac{a}{1-r}$.

   Due to the simple structure of the terms in a geometric series (the next term is r times the previous one), we can obtain explicit formulae for the partial sums and for the infinite sum itself. This is actually quite rare. For most convergent infinite series, it is far easier to show convergence than it is to find an explicit formula for the sum of the series.

6. **Absolute Convergence Test** (Thm. 2.7.6)

   If the infinite series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

This is a useful theorem to apply when considering series that have some negative terms. The converse is false, as shown by the alternating harmonic series. In fact, any *conditionally convergent* series would violate the converse.

7. **Alternating Series Test** (Thm. 2.7.7)

   Suppose that $\{a_n\}$ is a decreasing sequence of non-negative numbers that converges to 0. Then the alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

   This is the best test to apply when considering the convergence of an alternating series. By plotting the sequence of partial sums on a number line, it is easy to believe the veracity of this test. Note the difference between this test and the n-th term test.

8. **Cauchy Condensation Test** (Thm. 2.4.6)

   Suppose that $\{a_n\}$ is a decreasing sequence of non-negative numbers. The series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n a_{2^n} \;=\; a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots \tag{1}$$

   converges.

   This rather odd looking test for convergence is basically what we used to prove the harmonic series diverges. If $a_n = 1/n$, then the series in (1) reduces to $\displaystyle\sum_{n=1}^{\infty} 1$ which clearly diverges ($s_n = n$). This shows that the harmonic series diverges.

9. **p-series test** (Cor. 2.4.7)

   The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

   This is particularly useful when applying the comparison test. Note that the case on the border is when $p = 1$, which is the harmonic series. Once again, we see the importance of this series.

10. **Ratio Test** (Exercise 2.7.9)

    Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \neq 0 \; \forall n \in \mathbb{N}$. If

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \;=\; r \;<\; 1,$$

then the series converges absolutely. If the limit above is $r > 1$ (or infinite), the series diverges. If the limit above is $r = 1$, no conclusion can be drawn.

This useful test is surprisingly left to the exercise section in the textbook. You will prove the first part of the test on HW #5.

11. **Integral Test**

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n > 0 \; \forall n \in \mathbb{N}$. Let $f : [1, \infty) \to \mathbb{R}$ be the function obtained by replacing the n in the formula for a_n with the variable x. Suppose that f is a decreasing, continuous function with $\lim_{x \to \infty} f(x) = 0$. Then,

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{if and only if} \quad \int_1^{\infty} f(x)\, dx \text{ converges.}$$

This test is not given in the text for the simple reason that we have yet to define the Riemann integral. However, given its importance in the subject of infinite series, it is included here for reference purposes only. It gives a quick proof of the p-series test and another argument for the divergence of the harmonic series. However, you are **not** allowed to use it on your homework.