MATH 242 Principles of Analysis

Solution to Problem 2m in Section 3.1

2m. Prove that

$$\lim_{x\to 1} \frac{1}{5x-3} = \frac{1}{2}$$

using the ϵ-δ definition.

Proof: Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that

$$0 < |x-1| < \delta \implies |f(x) - \frac{1}{2}| < \epsilon.$$

Simplifying $|f(x) - L|$ gives

$$\left|\frac{1}{5x-3} - \frac{1}{2}\right| = \left|\frac{2-(5x-3)}{2(5x-3)}\right| = \left|\frac{5-5x}{2(5x-3)}\right| = \frac{5}{2}\cdot\frac{1}{|5x-3|}\cdot|x-1|$$

In order to make this smaller than ϵ, we need to bound the term $1/|5x-3|$. To do this we want the denominator to be bounded away from 0. Note that when $x = 3/5$ this term blows up to $+\infty$. This is a problem!

Suppose that $\delta \leq 1/5$, then $|x-1| < 1/5$ means $4/5 < x < 6/5$ and thus we have kept x away from the danger zone around $x = 3/5$. (Choosing $\delta \leq 1$ is not restrictive enough. This doesn't exclude $x = 3/5$.) Then, since $|5x-3|$ is smallest when $x = 4/5$, we have

$$\frac{4}{5} < x < \frac{6}{5} \implies \frac{1}{|5x-3|} < 1.$$

Then, letting $\delta = \min\{1/5, 2\epsilon/5\}$, we have

$$0 < |x-1| < \delta \implies |f(x) - \frac{1}{2}| = \frac{5}{2}\cdot\frac{1}{|5x-3|}\cdot|x-1| < \frac{5}{2}\cdot|x-1| < \frac{5}{2}\cdot\frac{2}{5\epsilon} = \epsilon$$

which completes the proof.