MATH 242 Principles of Analysis

Exam #2 Solutions **Prof. G. Roberts**

1. **(a)** Give a rigorous definition for $\lim_{x \to c+} f(x) = -\infty$.

   Answer: $\forall M > 0, \exists \delta > 0$ such that $f(x) < -M$ whenever $c < x < c + \delta$

   (b) State the Bolzano-Weierstrass Theorem.

   Answer: Every bounded sequence has a convergent subsequence.

2. Using the ϵ-δ definition for the limit of a function, prove that

$$\lim_{x \to 2} \frac{3}{2x-5} = -3 .$$

Answer: Fix $\epsilon > 0$. We compute that

$$|f(x) - L| = \left| \frac{3}{2x-5} - -3 \right| = \left| \frac{3 + 3(2x-5)}{2x-5} \right| = \left| \frac{6x-12}{2x-5} \right| = 6 \cdot \frac{1}{|2x-5|} \cdot |x-2|$$

We need to bound the term $1/|2x-5|$. Since x is approaching 2, it is crucial to pick a δ small enough so that $x \neq 5/2$, as the term $1/|2x-5|$ is infinite here. Suppose that $\delta \leq 1/4$. Then $|x-2| < \delta \Longrightarrow x \in (7/4, 9/4)$. The maximum of $1/|2x-5|$ occurs when the denominator is the **smallest**, which is when $x = 9/4$. Plugging in $x = 9/4$ yields a maximum of 2. Therefore, we have

$$6 \cdot \frac{1}{|2x-5|} \cdot |x-2| \;\leq\; 6 \cdot 2 \cdot |x-2|.$$

To make the last quantity less than ϵ, we require that $\delta \leq \epsilon/12$.

Choosing $\delta = \min\{1/4, \epsilon/12\}$, we have

$$0 < |x-2| < \delta \Longrightarrow |f(x) - L| = 6 \cdot \frac{1}{|2x-5|} \cdot |x-2| \leq 12|x-2| < 12 \cdot \epsilon/12 = \epsilon$$

which completes the proof.

3. Consider the following "proof" that $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$.

Proof: Applying the Big Limit Theorem for functions, we know that the limit of a product is equal to the product of the limits. Therefore, we obtain

$$\lim_{x \to 0} x^2 \cos \frac{1}{x} \;=\; \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \cos \frac{1}{x} \;=\; 0.$$

   (a) What's wrong with this "proof"?

   Answer: To apply the BLT, the limits of the individual functions must EXIST. Since

$$\lim_{x \to 0} \cos \frac{1}{x}$$

   does not exist (oscillating more and more as x approaches 0), we cannot apply the BLT.

(b) Give a correct proof using the ϵ-δ definition for the limit of a function.

Answer: Fix $\epsilon > 0$. Consider

$$|x^2 \cos\frac{1}{x} - 0| = |x^2 \cos\frac{1}{x}| = |x|^2 \cdot |\cos\frac{1}{x}| \leq |x|^2$$

since cosine of anything is always less than or equal to one. We want to have $|x|^2 < \epsilon$ which is equivalent to $|x| < \sqrt{\epsilon}$. Set $\delta = \sqrt{\epsilon}$. Then

$$0 < |x - 0| < \delta \Longrightarrow |f(x) - L| = |x|^2 \cdot |\cos\frac{1}{x}| \leq |x|^2 < \epsilon$$

which completes the proof.

4. Consider the function

$$f(x) = \begin{cases} |x+1| + |x-4| & \text{if } x \leq 2 \\ -x^2 + 4x + 1 & \text{if } x > 2 \end{cases}$$

(a) Is f continuous at $x = 2$? Explain, using the definition of continuity.

Answer: For f to be continuous at $x = 2$, we must show that

$$\lim_{x\to 2} f(x) = f(2) = 5.$$

Using right and left-hand limits, we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x + 1 + 4 - x = 5$$

since $|x+1| = x+1$ and $|x-4| = 4-x$ if x is near 2. Likewise, we have

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -x^2 + 4x + 1 = 5.$$

Since the left and right-hand limits are equal, and they equal the function value $f(2)$, we have shown that f is continuous at $x = 2$.

(b) Is f differentiable at $x = 2$? Explain, using the definition of the derivative.

Answer: Yes, f is differentiable at $x = 2$. Compute, using the definition of derivative, that

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{x + 1 + 4 - x - 5}{x - 2} = \lim_{x\to 2^-} \frac{0}{x - 2} = 0$$

and

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{-x^2 + 4x + 1 - 5}{x - 2} = \lim_{x\to 2^+} -(x - 2) = 0.$$

Since the left and right-hand limits are equal, it follows that the limit exists, f is differentiable at $x = 2$ and $f'(2) = 0$.

Note: No ϵ-δ arguments are required for this problem.

5. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, provide a **counterexample**.

(a) If $\{x_n\}$ and $\{y_n\}$ are each increasing sequences of *positive* numbers, then $\{x_n \cdot y_n\}$ is an increasing sequence.

Answer: TRUE

Since $\{x_n\}$ is increasing, we have $x_n \leq x_{n+1}$ for all n. We may multiply both sides of this inequality by y_n without changing the direction of the inequality because $y_n > 0$ for all n. Therefore,

$$x_n \, y_n \leq x_{n+1} \, y_n.$$

Since $\{y_n\}$ is increasing, we have $y_n \leq y_{n+1}$ for all n. We may multiply both sides of this inequality by x_{n+1} without changing the direction of the inequality because $x_n > 0$ for all n. This yields

$$x_{n+1} \, y_n \leq x_{n+1} \, y_{n+1}.$$

The transitivity of $\leq$ gives us

$$x_n \, y_n \leq x_{n+1} \, y_{n+1} \ \forall n$$

so that $\{x_n \cdot y_n\}$ is an increasing sequence.

(b) The sequence $\{x_n\}$ defined by $x_1 = 0.12$, $x_2 = 0.1212$, $x_3 = 0.121212$, $x_n = 0.12 \ldots 12$ (12 repeated n times) converges to $4/33$. *Hint:* Notice that $100x_{n+1} - 12 = x_n$.

Answer: TRUE

It is clear from the construction of the sequence that $\{x_n\}$ is increasing because at each step we are adding $12/10^{2n}$. It is equally clear that $\{x_n\}$ is bounded by say 1. By the Monotone Convergence Theorem, it follows that $\{x_n\}$ converges. Denote the limit of this sequence a. Using the hint and the Big Limit Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} (100x_{n+1} - 12) &= \lim_{n\to\infty} x_n \\
100 \lim_{n\to\infty} x_{n+1} - \lim_{n\to\infty} 12 &= a \\
100a - 12 &= a \\
a &= \frac{4}{33}
\end{aligned}$$

(c) Suppose that f and g are differentiable functions such that $g(f(x)) = e^{x^2}$, $f(0) = 2$ and $f'(0) \neq 0$. Then $g'(2) = 1$.

Answer: FALSE

Using the chain rule we have,

$$g'(f(x)) \cdot f'(x) = e^{x^2} \cdot 2x$$

and after plugging in $x = 0$, we see that

$$g'(f(0)) \cdot f'(0) = 0$$

or $g'(2) \cdot f'(0) = 0$. Since it is given that $f'(0) \neq 0$, it follows that $g'(2) = 0$.