MATH 242 Principles of Analysis

Exam #1 SOLUTIONS **October 11, 2002** **Prof. G. Roberts**

1. **(a)** Give the definition of the *greatest lower bound* of a set A.

   Ans.: We say that β is the greatest lower bound for A if and only if **(1)** β is a lower bound for A (ie. $\beta \leq x \ \ \forall x \in A$) and **(2)** $\beta \geq m$ for any lower bound m of the set A.

   (b) Suppose B is an infinite set. Give the definition for B being a *countable* set.

   Ans.: An infinite set B is countable if and only if there exists a bijection (1-1 and onto function) between B and the natural numbers $\mathbf{N}$.

   (15 pts.)

2. Let $A = \{x \in \mathbf{R} \mid |x-3| < 2\}$ and let B be the interval $(-1, 2]$. Let $f : \mathbf{R} \mapsto \mathbf{R}$ be given by $f(x) = e^x$.

   (a) What is the set $A - B$?

   Ans.: Using the definition of A, we compute that $A = (1, 5)$. Since $A - B$ is the set of all elements in A which are not in B, we have $A - B = (2, 5)$.

   Note: Be sure you understand interval notation. $\{1, \ldots, 5\}$ and $(1, 5)$ are completely different sets.

   (b) What is the set $f^{-1}(B)$?

   Ans.: This was the one "tricky" problem on the exam. Using the definition of f^{-1} of a SET, we have that

   $$f^{-1}(B) \ = \ \{x \in \mathbf{R} \mid f(x) \in B\}$$

   Thinking of the range of the exponential function, we know that all negative real numbers must be in $f^{-1}(B)$ because f sends these elements inside the interval $(0, 1)$ which is a subset of B. Since $f(\ln 2) = 2$, we conlcude that $f^{-1}(B) = (-\infty, \ln 2)$.

   (c) If $C = \{x^2 \mid x \in B\}$, find the lub(C) and the glb(C).

   Ans.: By considering a graph of $g(x) = x^2$ over the interval $(-1, 2]$, we see that the set C is the interval $[0, 4]$. Thus the lub$(C) = 4$ and the glb$(C) = 0$.

   (20 pts.)

3. For what value(s) of c does the equation

   $$|x+1| \ - \ |x-2| \ = \ c$$

   have an infinite number of solutions? State the solution in any such case. Explain. (12 pts.)

   Ans.: Using the geometric definition of absolute value, that $|a - b|$ equals the distance between a and b, any solution x to the above equation would satisfy:

   the difference of the distance between x and -1 and the distance between x and 2 equals c.

   The idea is to consider the three regions of the number line determined by the key points -1 and 2, and then evaluate the left-hand side of the above equation. For example, for $x \leq -1$, we obtain the equation

   $$-(x+1) - -(x-2) = c$$

which reduces to $c = -3$. This means that any x in this region will satisfy the equation as long as $c = -3$. There will be an infinite number of solutions to the equation if $c = -3$. In this case, the solution is $(-\infty, -1]$.

Another way to see this is to think geometrically. Picking an x in this first region and looking at the difference of the distances will always give the negative of the distance between -1 and 2.

Similarly, by choosing x from the region $x \geq 2$, we obtain $x + 1 - (x - 2) = c$ or $c = 3$. Again, there are an infinite number of solutions, given by the interval $[2, \infty)$, which satisfy the equation when $c = 3$. This is also easy to see geometrically.

Finally, in the middle region, $-1 < x < 2$, we find that there is a **unique** solution to the equation given by $x = (c + 1)/2$. To obtain a solution in this region we need to choose c such that $-3 < c < 3$. In this case, there will be only one solution.

Thus the only way to have an infinite set of solutions is to choose $c = 3$ or $c = -3$. A slightly tougher problem is to ask which c values yield NO solutions.

4. Prove that the set of real numbers is an uncountable set. (10 pts.)

   Ans.:

   It suffices to show that the interval $(0, 1)$ is not countable. Since $(0, 1)$ is a subset of $\mathbf{R}$, it follows that $\mathbf{R}$ is not countable. (If $\mathbf{R}$ were countable, then any subset of $\mathbf{R}$ would also be countable, yielding a contradiction.)

   We use Cantor's diagonalization proof. If the interval $(0, 1)$ is countable, there exists a function $f : \mathbf{N} \to (0, 1)$ which is both one-to-one and onto. This means we can list all the real numbers in $(0, 1)$ as $\{x_1, x_2, x_3, x_4, \ldots\}$. Put each number in its base 10 decimal expansion and denote the jth decimal place of the element x_i by x_{ij}. So our list might look something like this:

$$\begin{aligned} x_1 &= \quad 0.3418 \ldots \\ x_2 &= \quad 0.4106 \ldots \\ x_3 &= \quad 0.1119 \ldots \\ x_4 &= \quad 0.3244 \ldots \\ x_5 &= \quad \ldots \end{aligned}$$

   Every real number in (0,1) is supposed to appear somewhere on the right-hand side of this list, since f is onto. However, and this was Cantor's wonderful insight, we can produce a number $y \in (0, 1)$ which **cannot** be on the list. We will construct the decimal places of y using the following formula. For notation, let b_k be the kth decimal place of y, so $y = 0.b_1 b_2 b_3 b_4 \ldots$. Now define

$$b_k = \begin{cases} 4 & \text{if } x_{kk} \neq 4 \\ 5 & \text{if } x_{kk} = 4 \end{cases}$$

   So in the example above, y will begin as $y = 0.4445\ldots$. No matter what follows in the list of x's, the number y will never appear on the right-hand side of the list, since the kth decimal place of y is never equal to the kth decimal place of x_k. However, y is certainly a real number between 0 and 1. Thus the function f is not onto which is a contradiction. QED

5. Find the $\lim\limits_{n\to\infty} x_n$ for the sequence

$$x_n = \frac{\sin(2n)}{n^2} + 4$$

and prove that $\{x_n\}$ converges to your limit using the ϵ-n_0 definition of convergence. (15 pts.)

Ans.: Using Calculus, we see that the limit is 4 since the first term is dominated by $1/n^2$ and therefore converging to 0. We then have, using the definition of convergence,

$$|x_n - a| = \left|\frac{\sin(2n)}{n^2} + 4 - 4\right| = \left|\frac{\sin(2n)}{n^2}\right| \leq \frac{1}{n^2}.$$

We want this quantity to be small.

Fix $\epsilon > 0$. We have that

$$\frac{1}{n^2} < \epsilon \text{ iff } n^2 > \frac{1}{\epsilon} \text{ iff } n > \frac{1}{\sqrt{\epsilon}}.$$

Choose n_0 to be any natural number greater than $\frac{1}{\sqrt{\epsilon}}$. Then

$$|x_n - a| = \left|\frac{\sin(2n)}{n^2}\right| \leq \frac{1}{n^2} < \epsilon \quad \forall n \geq n_0.$$

QED

Note that we could have chosen n_0 to be any natural number greater than $\frac{1}{\epsilon}$ (thus avoiding the square roots all together) because we have

$$\frac{1}{n^2} < \frac{1}{n} < \epsilon \quad \forall n \geq n_0.$$

6. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, provide a **counterexample**.

(a) The number $\sqrt{6}$ is irrational.

Ans.: TRUE. Use proof by contradiction. Suppose that $\sqrt{6}$ is rational and write $\sqrt{6} = m/n$ with m and n integers sharing no common factors. Squaring both sides and simplifying yields $6n^2 = m^2$ or

$$2 \cdot 3 \cdot n^2 = m^2$$

By the uniqueness of prime factorization, we know that the number of prime factors must be the same for both numbers $2 \cdot 3 \cdot n^2$ and m^2. However, this is impossible because the number on the left has an odd number of 2's and an odd number of 3's while the number on the right-hand side has an even number of 2's and an even number of 3's. (A number squared always has an even number of any particular prime.) This contradiction completes the proof.

(b) If $a \leq b$ and $c \leq d$, then $ac \leq bd$.

Ans.: FALSE. Let $a = -3$, $b = -2$, $c = -1$ and $d = 0$. Then we have $a \leq b$ and $c \leq d$ but $ac = 3 > 0 = bd$.

(c) $3 + 7 + 11 + 15 + \ldots + 4n - 1 = n(2n+1) \quad \forall n \in \mathbf{N}$

Ans.: TRUE. Use proof by induction.

Base Case: $(n = 1)$

We have $3 = 1(2 \cdot 1 + 1)$ is true so the base case is satisfied.

Inductive Step: Assume the statement P_k is true. Show that P_{k+1} is true.

In other words, assume

$$3 + 7 + 11 + 15 + \ldots + 4k - 1 \; = \; k(2k+1)$$

is true and show that

$$3 + 7 + 11 + 15 + \ldots + 4k - 1 + 4(k+1) - 1 \; = \; (k+1)(2(k+1)+1)$$

is true. Using the inductive hypothesis, we have that

$$\begin{aligned}
3 + 7 + 11 + 15 + \ldots + 4k - 1 + 4(k+1) - 1 \; &= \; k(2k+1) + 4(k+1) - 1 \\
&= \; 2k^2 + k + 4k + 4 - 1 \\
&= \; 2k^2 + 5k + 3 \\
&= \; (2k+3)(k+1) \\
&= \; (k+1)(2(k+1)+1)
\end{aligned}$$

which finishes the proof.

(30 pts.)