Principles of Analysis: March 1, 2002
Introduction to Discrete Dynamical Systems

In Section 2.4 of Bilodeau and Thie's text, we began to study sequences defined recursively. These sequences were specially chosen to build on previous material in the text (like the Monotone Convergence Theorem). Most of the examples were monotonic and it was not hard to calculate their limits. However, recursive sequences and in general, the notion of **iterating** a function, is an important and often complicated field in its own right, known as **discrete dynamical systems**. This is part of the subject more popularly referred to as **chaos theory**. Applications in this area are plentiful, including everything from climate prediction to flying to the moon for next to nothing. This worksheet is designed to give you a brief introduction to this fascinating subject.

1 Two Simple Examples

Let's begin with a simple example of a dynamical system. Consider the equation

$$x_{n+1} = \frac{1}{2}x_n + 1, \; x_0 = c$$

which is known as a **difference equation**. Here we are using the constant c to represent the initial condition or **initial seed** of the sequence. (Note that we are starting with x_0 not x_1, as we normally do with sequences, for notational reasons which will be made clear later.) Given an initial seed c, we refer to the sequence

$$c, \, x_1, \, x_2, \, x_3, \, \ldots$$

generated by the recursion formula as the **orbit** of c. Starting with different c-values will lead to different sequences, so different initial seeds have different orbits. However, no matter what c-value we choose, all of the orbits for this difference equation will share at least one property in common. Let's find this property.

1. Using the dynamical system above, set $c = 10$. That is, start the sequence with $x_0 = 10$. Compute the first ten terms of this sequence. Does it converge? Find the limit. (No proofs required.)

2. Using the same system, now try $c = -4$ as the initial seed. Compute the first ten terms of this sequence. Does it converge? Find the limit and compare with the previous question. What do you notice? How is this sequence different from that of the previous question?

3. Let a be the limit from the previous questions. What is the sequence obtained if we set $x_0 = a$?

4. TRUE or FALSE: No matter what seed $x_0 = c$ we start with, the corresponding sequence which is generated always converges to the same limit a. Can you prove or disprove this?

   In dynamical systems terminology, we say that $x = 2$ is a **fixed point** of the system because plugging 2 into the system gives back 2. (So the point $x = 2$ is fixed by the system.) We also call $x = 2$ an **attracting fixed point** because all the initial seeds nearby converge to $x = 2$ under iteration. In fact, since every initial seed converges to $x = 2$, it is a **global attractor**.

Now consider a similar dynamical system given by

$$x_{n+1} = 2\,x_n + 1,\;\; x_0 = c.$$

Here we have replaced the 1/2 from the previous difference equation with 2. This will turn out to be very important.

5. Using this new system, set $c = 10$. Compute the first five terms of this sequence. Does it converge? What happens?

6. Using the new system, now try $c = -4$ as the initial seed. Compute the first five terms of this sequence. Does it converge? How is the sequence from this question different from that of the previous question?

7. TRUE or FALSE: No matter what seed $x_0 = c$ we start with, the corresponding sequence which is generated always heads to ∞ or $-\infty$. Can you prove or disprove this?

2 Iteration of Functions

There is an alternative manner of approaching the above problems which leads to some very useful geometric and analytic ideas. Instead of using a difference equation, we can think of a discrete dynamical system as **iteration** of a given function. Instead of a recursive formula, we take a function $f(x)$ and plug in our initial seed c. Out comes $f(c)$. This in turn is plugged in to the function, and out comes $f(f(c))$. Continuing the process, we obtain the successive iterates of c, that is, the orbit of c is given by

$$c,\; f(c),\; f(f(c)),\; f(f(f(c))),\; \ldots$$

Our first difference equation above, $x_{n+1} = (1/2)x_n + 1$, is equivalent to iteration of the function

$$f(x) = \frac{1}{2}\,x + 1$$

while the second difference equation we studied, $x_{n+1} = 2x_n + 1$, is the same dynamical system as iteration of the function

$$g(x) = 2x + 1\,.$$

Notation: The nth iterate of a function f is usually denoted f^n. So for example, if $g(x) = 2x + 1$, then

$$g^2(x) = g(g(x)) = g(2x + 1) = 4x + 3,$$

and similarly

$$g^3(x) = g(g^2(x)) = g(4x + 3) = 8x + 7.$$

In general, (by induction), we have

$$g^n(x) = 2^n\,x + 2^n - 1$$

from which it is easy to see why successive iterates of our second difference equation head off to $\pm\infty$. Note however, that $g^n(-1) = -1\ \forall n$. We call $x = -1$ a **repelling fixed point** because it is fixed by the system, $g(-1) = -1$, and because all nearby initial seeds are repelled away from $x = -1$. Repelling fixed points can be hard to find because everybody is heading away from them, whereas attracting fixed points are easy to find because lots of orbits are converging toward them.

The general goal in studying a particular dynamical system is to describe all of the possible orbits of the system. Sometimes this is very easy to do (as in our two previous examples), but sometimes this is quite challenging, even for seemingly simple functions (such as quadratics).

There are various kinds of orbits possible. You have already seen an example of one of them, a fixed point. It is also possible to have a **periodic orbit** or **cycle** of a given period. Consider the following examples.

8. Suppose $f(x) = x^2 - 1$. What is the orbit of the initial seed $x_0 = 0$ under f? What is the orbit of the initial seed $x_0 = \sqrt{2}$? What is the orbit of the initial seed $x_0 = 0.5$?

9. Consider the function $g(x) = (-3/2)x^2 + (5/2)x + 1$. What is the orbit of the initial seed $x_0 = 0$ under g?

In general, we say that a point is on a **period n cycle** if it returns to its initial position after n iterates, and n is the smallest such natural number for which this happens. (If an orbit repeats every 4 iterates, then it will also repeat every 8 iterates, every 12 iterates and so on. We call 4 the period if this is the smallest such integer for which the orbit repeats itself.)

3 Computers and Chaos

Most orbits of a dynamical system, however, are usually not periodic. They may head towards an attracting fixed point, or attracting periodic cycle. They may head off towards ∞ or $-\infty$. But they also may never do anything predictable, bouncing around an interval forever, seemingly never wanting to stay anywhere, and coming arbitrarily close to any point in the interval. These orbits are often described as **chaotic** and they are one of the features which makes the subject exciting.

To visualize such types of systems we can make use of computers. In fact, the subject of dynamical systems flourished with the growth of computer technology. Computers allowed researchers to quickly investigate orbits of a given dynamical system. Fascinating images called **fractals** were generated and the popularity of the subject immediately followed.

Consider the following 2-dimensional dynamical system, known as the standard map. Here there are two equations and two initial seeds θ_0 and y_0. The domain of this system is actually a cylinder (not the xy-plane), so the θ coordinate is restricted to $-\pi \leq \theta \leq \pi$, while the y-coordinate can be any real number. Here, k is called a **parameter**, which we can vary to see how our dynamical system changes. The standard map is given by

$$\begin{aligned} \theta_{n+1} &= \theta_n + y_{n+1} \\ y_{n+1} &= y_n - k\sin(2\pi\theta_n)/(2\pi) \end{aligned}$$

A laptop demonstration will be given once you reach this point.

4 Web Diagrams and the Logistic Map

There is an important geometric construct which will enable us to analyze the behavior of a dynamical system without having to compute by hand a bunch of individual orbits. This is known as a **web diagram**. The best way to learn this is via examples and graphs.

A blackboard demonstration will be given once you reach this point.

One important dynamical system which has generated tons of research and interest is the **Logistic Map**. It is one of the first models studied in Biology for predicting populations where there is some growth rate k of the population and a "carrying capacity" N for the size of the population. If the population reaches this size N, it is doomed, and dies out immediately. For simplicity, we will take $N = 1$ and treat k as a parameter. The model is then given by the equation

$$L_k(x) \;=\; k\,x(1-x)$$

where k is a positive real number, and x is restricted to the interval $0 \leq x \leq 1$.

We will use the computer to help understand this system, but first try a few examples to get a feel for this dynamical system.

10. What are the orbits of $x = 0$ and $x = 1$ under the logistic map for any k? Explain this physically in terms of the population.

11. Draw the graph of $L_k(x)$ for the k-values $k = 1, 2, 3, 4$ on the same axes. How does k effect the graphs?

12. What are the fixed points for this system? One of your answers should depend on k. Are the fixed points always in the interval $[0, 1]$? Explain.

13. Can you tell when your fixed points are attracting or repelling? What is the key feature of the function which provides this information?

14. Can you tell if the logistic map will have any period 2 points? What must k be in order for this to happen? What does this mean physically for the population?

15. Do you think it is possible to get chaotic orbits for any values of k? If so, which ones?

    There are many, many questions to study here and many fascinating results to investigate. One could easily write a thesis on this family (in fact, many people have), so we'll now use the computer to get an idea of the behavior of the logistic map.

    A laptop demonstration will be given once you reach this point.

THE END or THE BEGINNING?