Principles of Analysis: Sept. 11, 2002
Axioms of the Real Number System

The set of real numbers, denoted $\mathbf{R}$, is a collection of numbers on which two **binary operations** are defined, denoted by "+" (addition) and "·" (multiplication). A binary operation on a set A, is a function which maps an ordered pair (x, y) of elements in A to another element in A. The key feature to a binary operation is **closure**, that is, the result of the operation must stay in the same set you started with. For example, the set of negative real numbers is not closed under multiplication because a product of two negatives gives a positive number, therefore outside the original set.

The following **axioms** are *assumed* to hold regarding the operations of addition and multiplication of real numbers. For all $x, y, z \in \mathbf{R}$:

Axiom 1	Commutative Property	$x + y = y + x, \quad x \cdot y = y \cdot x$
Axiom 2	Associative Property	$x + (y + z) = (x + y) + z, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Axiom 3	Distributive Property	$x \cdot (y + z) = x \cdot y + x \cdot z$

Axiom 4	Identity element of addition	There is a unique number denoted by "0" for which $x + 0 = x$ for all x in $\mathbf{R}$.
	Identity element of multiplication	There is a unique number denoted by "1" with $1 \neq 0$ for which $x \cdot 1 = x$ for all x in $\mathbf{R}$.
Axiom 5	Additive inverse	For each x in $\mathbf{R}$, there is a unique number y such that $x + y = 0$. y is denoted as $-x$.
	Multiplicative inverse	For each $x \neq 0$ in $\mathbf{R}$, there is a unique number y such that $x \cdot y = 1$. y is denoted as x^{-1}.

These five axioms are often referred to as the **field axioms**. To complete the description of the real numbers as we know them, we need four **order axioms** as well.

Axiom 6	For all $x, y \in \mathbf{R}$, one and only one of the following is true: $x < y, \quad y < x \quad$ or $\quad x = y$.
Axiom 7	If $x < y$ and $y < z$, then $x < z$. ($<$ is a transitive relation.)
Axiom 8	If $x < y$, then $x + z < y + z$ for every $z \in \mathbf{R}$.
Axiom 9	If $x < y$ and $0 < z$, then $x \cdot z < y \cdot z$. (Note that z is positive here.)

This gives a definition for the "less than" operation $<$. We can define the "greater than" operation $>$ by

$$x > y \quad \text{if and only if} \quad y < x.$$

A number x is **positive** if $x > 0$ and a number x is **negative** if $x < 0$.

From the axioms above we can prove rigorously the basic properties of the real number system we use on a regular basis. However, you must **assume nothing** other than the above axioms. At times this will be frustrating and seem overly pedantic. However, the goal is for you to gain an appreciation for the foundations of the real number system as well as develop your skills at proving theorems.

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1. Carefully justify each step with the appropriate axiom to prove the following:

$$\text{If } x+y=x+z, \text{ then } y=z.$$

   This is called the **Cancellation Law of Addition**.

   Proof:

$$\begin{aligned} y &= y+0 \\ &= y+(x+(-x)) \\ &= (y+x)+(-x) \\ &= (x+y)+(-x) \\ &= (x+z)+(-x) \\ &= (z+x)+(-x) \\ &= z+(x+(-x)) \\ &= z+0 \\ &= z \end{aligned}$$

   Note: This proves the uniqueness of the additive inverse, since if $x+y=0$ and $x+z=0$, we have $x+y=x+z$ which implies $y=z$. Thus, $-x$ is unique.

2. Generalize the argument from the previous problem to prove:

$$\text{If } x \cdot y = x \cdot z \text{ and } x \neq 0, \text{then } y=z.$$

   Where does the proof fall apart if $x=0$? Use this result to prove the multiplicative inverse x^{-1} is unique.

3. Prove that $x \cdot 0 = 0$ for any $x \in \mathbf{R}$. (Yes, we must prove this!) *Hint:* Start with the expression $x \cdot 0 + x \cdot 0$ and simplify using only the axioms above. The Cancellation Law of Addition may be helpful.

4. Prove that $-x = (-1) \cdot x$ for any $x \in \mathbf{R}$. The term on the left-hand side of the equation is the element which is the additive inverse of x. The term on the right-hand side of the equation is the product of the number -1 and x. You must prove these are equivalent. *Hint:* Substitute $(1+(-1))$ in for 0 in the equation $x \cdot 0 = 0$. (Which 0 should you replace?) Simplify and use the uniqueness of the additive inverse.