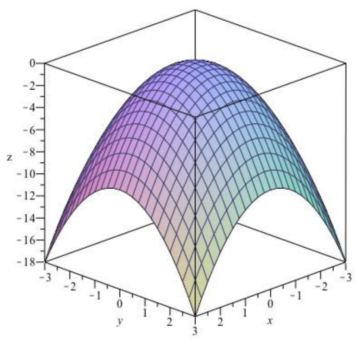
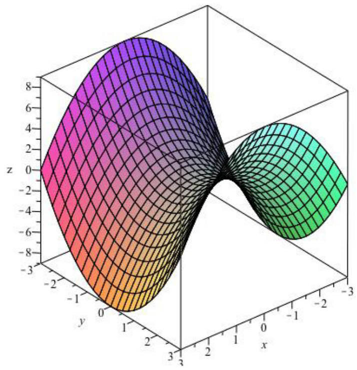
MATH 241-02 Multivariable Calculus

Exam #1 SOLUTIONS **February 20, 2019** **Prof. G. Roberts**

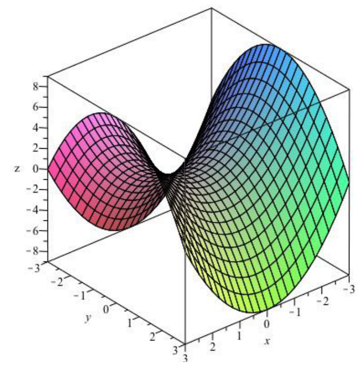
1. Match the graph of each quadric surface **(a)**–**(f)** with the correct equation **(i)**–**(vi)**. There is exactly **one** equation for each graph. Provide a brief explanation for your choices. (18 pts.)



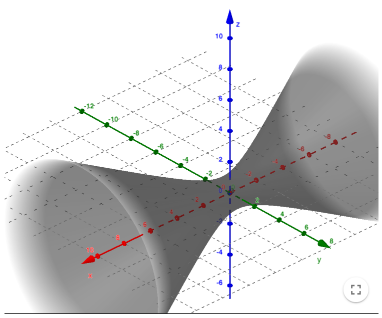
(a)



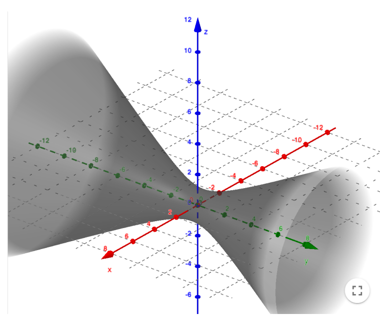
(b)



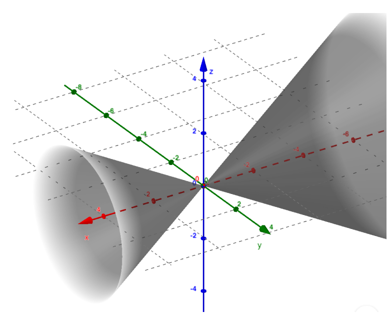
(c)



(d)



(e)



(f)

(i) $z = x^2 - y^2$ **(ii)** $x^2 - \dfrac{y^2}{3} + z^2 = 1$ **(iii)** $z = -x^2 - y^2$

(iv) $z = y^2 - x^2$ **(v)** $-\dfrac{x^2}{3} + y^2 + z^2 = 1$ **(vi)** $-\dfrac{x^2}{3} + y^2 + z^2 = 0$

(a) **(iii)** **(b)** **(iv)** **(c)** **(i)**

(d) **(v)** **(e)** **(ii)** **(f)** **(vi)**

Answer: Graph **(a)** is an upside down bowl, which corresponds to $z = -(x^2 + y^2) = -x^2 - y^2$. Graphs **(b)** and **(c)** are saddles. In graph **(b)**, the cross sections for $x = k$ are parabolas opening up while for $y = k$, they are parabolas opening down. This agrees with the equation $z = y^2 - x^2$. In graph **(c)** the parabolas open up for the traces with $y = k$ and open down when $x = k$. This agrees with the equation $z = x^2 - y^2$. Graphs **(d)** and **(e)** are hyperboloids of one sheet. The cross sections $x = k$ for **(d)** are circles, which agrees with the equation $-\frac{x^2}{3} + y^2 + z^2 = 1$, while graph **(e)** has circular cross sections for the traces $y = k$, which agrees with the equation $x^2 - \frac{y^2}{3} + z^2 = 1$. Finally, graph **(f)** is a cone symmetric with respect to the x-axis, which matches the remaining equation $-\frac{x^2}{3} + y^2 + z^2 = 0$.

2. Multiple Choice: Choose the best answer available (no work required). (4 pts. each)

   (a) Which equation represents a sphere with center $(-1, 0, 2)$ passing through the point $(3, 1, -1)$?

   (i) $x^2 + y^2 + z^2 = 11$,
   (ii) $x^2 + 2x + y^2 + z^2 + 4z = 13$,
   (iii) $x^2 - 2x + y^2 + z^2 + 4z = 21$,
   (iv) $x^2 + 2x + y^2 + z^2 - 4z = 21$,
   (v) $x^2 + 2x + y^2 + z^2 - 4z = 26$.

   Answer: (iv). The radius of the sphere is the distance between $(-1, 0, 2)$ and $(3, 1, -1)$, which is $\sqrt{(3 - -1)^2 + (1 - 0)^2 + (-1 - 2)^2} = \sqrt{26}$. Therefore, the equation of the sphere is $(x + 1)^2 + y^2 + (z - 2)^2 = 26$, which simplifies to $x^2 + 2x + y^2 + z^2 - 4z = 21$.

   (b) Suppose that $\mathbf{v}$ and $\mathbf{w}$ are two non-parallel vectors in $\mathbb{R}^3$. Which of the following statements is **NOT** true?

   (i) $\mathbf{v} \times \mathbf{w}$ is a vector.
   (ii) $\mathbf{v} \times \mathbf{v} = 0$.
   (iii) $\mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{v}$.
   (iv) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.
   (v) $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

   Answer: (iii). The cross product is anti-commutative because of the right-hand rule, so $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$. The equation $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ simply says that $\mathbf{w}$ is orthogonal to the cross product of $\mathbf{v}$ and $\mathbf{w}$, which is true by definition of the cross product.

   (c) The traces of the quadric surface $4x^2 - y^2 - z^2 = 1$ in the planes $z = k$ are

   (i) circles,
   (ii) ellipses,
   (iii) parabolas,
   (iv) hyperbolas,
   (v) two intersecting lines.

   Answer: (iv). Taking cross sections $z = k$ yields $4x^2 - y^2 = 1 + k^2$, which means the traces are hyperbolas "munching" the x-axis.

   (d) The path of $\mathbf{r}(t) = \cos 3t\,\mathbf{i} + 4\,\mathbf{j} + \sin 3t\,\mathbf{k}$ traces out a

   (i) helix (slinky) in xyz-space,
   (ii) line in xyz-space,
   (iii) circle of radius 3 in the plane $x + 4y + z = 0$,
   (iv) circle of radius 3 in the plane $y = 4$,
   (v) circle of radius 1 in the plane $y = 4$.

   Answer: (v). This one was a little tricky. The y-component of the curve is just 4 (not $4t$). Consequently, the curve lies in the plane $y = 4$. Since $(x(t))^2 + (z(t))^2 = \cos^2(3t) + \sin^2(3t) = 1$ for all t, the curve is a circle of radius 1.

3. Given the vectors $\mathbf{v} = \mathbf{i} + 7\,\mathbf{k}$ and $\mathbf{w} = -2\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$, compute each of the following quantities: (16 pts.)

(a) 4**v** - 3**w**

Answer: $4\mathbf{v} - 3\mathbf{w} = 4(\mathbf{i} + 7\,\mathbf{k}) - 3(-2\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}) = 4\,\mathbf{i} + 28\,\mathbf{k} + 6\,\mathbf{i} - 9\,\mathbf{j} - 6\,\mathbf{k} = 10\,\mathbf{i} - 9\,\mathbf{j} + 22\,\mathbf{k}$.

(b) $|\mathbf{v} - \mathbf{w}|$

Answer: $|\mathbf{v} - \mathbf{w}| = |3\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}| = \sqrt{9 + 9 + 25} = \sqrt{43}\,.$

(c) $\text{proj}_{\mathbf{v}}\mathbf{w}$ (the vector projection of **w** onto **v**)

Answer: Using $\text{proj}_{\mathbf{v}}\mathbf{w} = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$, we have

$$\text{proj}_{\mathbf{v}}\mathbf{w} = \left(\frac{-2 + 0 + 14}{1 + 49}\right)(\mathbf{i} + 7\,\mathbf{k}) = \frac{12}{50}\,\mathbf{i} + \frac{84}{50}\,\mathbf{k} = \frac{6}{25}\,\mathbf{i} + \frac{42}{25}\,\mathbf{k}.$$

(d) $\mathbf{v} \times \mathbf{w}$

Answer: $\mathbf{v} \times \mathbf{w} =$

$$\left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 1 & 0 & 7 & 1 & 0 \\ -2 & 3 & 2 & -2 & 3 \end{array}\right| = 0\,\mathbf{i} - 14\,\mathbf{j} + 3\,\mathbf{k} - (0\,\mathbf{k} + 21\,\mathbf{i} + 2\,\mathbf{j}) = -21\,\mathbf{i} - 16\,\mathbf{j} + 3\,\mathbf{k}.$$

Remember that you can check the cross product by confirming that the dot product between the resulting vector and each of the original vectors is zero.

4. Below are equations for two different pairs of planes. For one pair of equations, the planes are parallel; for the other pair of equations, the planes intersect. Identify which pair is which, and find the **acute** angle (to the nearest degree) between the two planes that intersect. (10 pts.)

$$\textbf{(i)}\quad \begin{aligned} x - y - 2z &= 10 \\ 4x + 2y - 4z &= 15 \end{aligned} \qquad\qquad \textbf{(ii)}\quad \begin{aligned} x - y - 2z &= 10 \\ 5x - 5y - 10z &= 13 \end{aligned}$$

Answer: The key is to consider the normal vectors for each plane. For the first pair of planes, we have $\mathbf{n_1} =< 1, -1, -2 >$ and $\mathbf{n_2} =< 4, 2, -4 >$. Since these two vectors are not scalar multiples of each other, they are not parallel, and thus the two planes are not parallel (and must intersect in a line.) On the other hand, the second pair of planes has $\mathbf{n_1} =< 1, -1, -2 >$ and $\mathbf{n_2} =< 5, -5, -10 >$. Since $\mathbf{n_2} = 5\mathbf{n_1}$, the normal vectors are parallel and thus the two planes are parallel (they are not the same plane because $13 \neq 5 \cdot 10$.)

To find the acute angle of intersection between the first pair of planes, we find the angle of intersection of the normals (by definition). Using the dot product formula $\mathbf{n_1} \cdot \mathbf{n_2} = |\mathbf{n_1}||\mathbf{n_2}| \cos\theta$, we find $4 - 2 + 8 = \sqrt{6} \cdot \sqrt{36}\,\cos\theta$, which implies

$$\cos\theta = \frac{10}{6\sqrt{6}} = \frac{5}{3\sqrt{6}}\,.$$

Thus, $\theta = \cos^{-1}(5/3\sqrt{6}\,) \approx 47°$.

5. Consider the three points $P = (1, 0, 3)$, $Q = (-1, 2, 2)$, and $R = (5, -1, 2)$. (14 pts.)

(a) Find the equation of the plane containing all three points.

Answer: To find the equation of the plane, we first need a normal vector to the plane. This can be obtained by taking the cross product of any two vectors in the plane. The vectors $\overrightarrow{PQ} =< -2, 2, -1 >$ and $\overrightarrow{PR} =< 4, -1, -1 >$ are each in the plane. The vector $\overrightarrow{PQ}$ is found by subtracting the coordinates of P from the coordinates of Q (i.e., $\overrightarrow{PQ} = Q - P$). The cross product is then computed as

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left| \begin{array}{ccc:cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ -2 & 2 & -1 & -2 & 2 \\ 4 & -1 & -1 & 4 & -1 \end{array} \right| = -2\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k} - (8\,\mathbf{k} + \mathbf{i} + 2\,\mathbf{j}) = -3\,\mathbf{i} - 6\,\mathbf{j} - 6\,\mathbf{k}.$$

Thus, a normal vector to the plane is $\mathbf{n} =< -3, -6, -6 >$. This means the plane has the equation $-3x - 6y - 6z = d$, where d is some constant. To find d we can plug in any of the three given points. Using point P (the easiest choice), we find that $-3(1) - 6(3) = d$, so $d = -21$. Dividing both sides by -3, the equation of the plane is $x + 2y + 2z = 7$. Note that all three points satisfy this equation.

(b) Find the area of the triangle PQR.

Answer: The area of triangle PQR is

$$\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\,|< -3, -6, -6 >| = \frac{1}{2}\sqrt{9 + 36 + 36} = \frac{9}{2}.$$

6. Find the point(s) where the line passing through $P(0, \frac{3}{2}, 1)$ and $Q(2, -\frac{1}{2}, -3)$ intersects the surface $2x^2 + 4y^2 = 3z^2$. (10 pts.)

Answer: The first step is to find the equation of the line between the points P and Q. The direction of this line is $\overrightarrow{PQ} =< 2, -2, -4 >$. Recall that the parametric equation for a line with direction $\mathbf{v}$ and passing through the point $\mathbf{r}_0$ is $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. Thus, using the point P as our chosen point, the equation of the line is given by $x = 2t, y = \frac{3}{2} - 2t, z = 1 - 4t$, for $t \in \mathbb{R}$.

To determine where the line intersects the surface $2x^2 + 4y^2 = 3z^2$ (a cone), we substitute the equation of the line into this equation and solve for t. We have

$$\begin{aligned} 2(2t)^2 + 4\left(\frac{3}{2} - 2t\right)^2 = 3(1 - 4t)^2 \implies 8t^2 + 4\left(\frac{9}{4} - 6t + 4t^2\right) &= 3(1 - 8t + 16t^2) \\ \implies 8t^2 + 9 - 24t + 16t^2 &= 3 - 24t + 48t^2 \\ \implies -24t^2 = -6 \quad \text{or} \quad t^2 &= \frac{1}{4}. \end{aligned}$$

Therefore $t = \pm\frac{1}{2}$. Plugging these values back into the equation of the line gives the points $(1, \frac{1}{2}, -1)$ and $(-1, \frac{5}{2}, 3)$. Notice that each point lies on the cone since it satisfies the equation $2x^2 + 4y^2 = 3z^2$.

7. Consider the vector function $\mathbf{r}(t) = <e^{2t}, \sqrt{1-2t}\,, t\sin t>$. (16 pts.)

(a) Find the domain of the vector function.

Answer: The domain of the vector function is the intersection of the domains of each component. The first and third component are defined for all real numbers, consequently, the domain of the function is simply the domain of the second component. In order for the square root to be defined, we must have $1 - 2t \geq 0$, which simplifies to $2t \leq 1$ or $t \leq 1/2$. Thus the domain of the function is $(-\infty, 1/2]$ or $t \leq 1/2$.

(b) Find the unit tangent vector $\mathbf{T}(t)$ at the point where $t = 0$.

Answer: To find the unit tangent vector we first compute $\mathbf{r}'(t)$. Using the chain rule and the product rule, we have

$$\mathbf{r}'(t) = <2e^{2t}, \frac{1}{2}(1-2t)^{-1/2} \cdot -2, \sin t + t\cos t> = <2e^{2t}, \frac{-1}{\sqrt{1-2t}}, \sin t + t\cos t> .$$

To find the tangent vector at the point where $t = 0$, we first plug in $t = 0$ to obtain $\mathbf{r}'(0) =< 2, -1, 0 >$. Then, to make this vector a unit vector, we divide by its length, $|\mathbf{r}'(0)| = \sqrt{4+1} = \sqrt{5}$. Hence,

$$\mathbf{T}(0) = \frac{1}{\sqrt{5}} <2, -1, 0> = <\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0> .$$

(c) Find parametric equations for the tangent line to the curve parametrized by $\mathbf{r}(t)$ at the point where $t = 0$.

Answer: To find parametric equations of a line we need a point and a direction. The point is found by plugging $t = 0$ into the function $\mathbf{r}(t)$ yielding $\mathbf{r}_0 = (1, 1, 0)$. To find the direction, we can use the derivative $\mathbf{r}'(0) = <2, -1, 0>$ or the unit tangent vector $\mathbf{T}(0) = <\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0>$. Actually, any scalar multiple of either vector would suffice. Thus, parametric equations describing the tangent line are

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 - t \\ z &= 0\,, \end{aligned}$$

or

$$\begin{aligned} x &= 1 + \frac{2}{\sqrt{5}}t \\ y &= 1 - \frac{1}{\sqrt{5}}t \\ z &= 0\,. \end{aligned}$$

Note that the tangent line lies in the xy-plane.