Tangent Planes

Recall from Calc 1 that the equation of the tangent line to the function \( y = f(x) \) at the point \((x_0, y_0)\) is given by
\[
y - y_0 = f'(x_0)(x - x_0).
\]
This is the point-slope form of a line with slope \( m = f'(x_0) \), passing through the point \((x_0, y_0)\). If \( f \) is differentiable at \( x_0 \), then as we zoom in on the graph of \( f \) near \((x_0, y_0)\), the graph looks more and more like the tangent line.

We now generalize this same idea to a function of two variables, \( z = f(x, y) \). Suppose that \((x_0, y_0, z_0)\) is a point on the graph of \( f \), that is, \( z_0 = f(x_0, y_0) \), and suppose that both first partial derivatives \( f_x \) and \( f_y \) exist and are continuous at \((x_0, y_0)\). Then the tangent plane to the graph of \( f \) at \((x_0, y_0, z_0)\) is given by
\[
z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]
(1)
The idea here is to capture the change in the function \( f \) in both the \( x \)-direction, through the term \( f_x(x_0, y_0)(x - x_0) \), and the \( y \)-direction, via the term \( f_y(x_0, y_0)(y - y_0) \). Notice that equation (1) is of the form \( ax + by + cz = d \), so that it represents the equation of a plane, and that \((x_0, y_0, z_0)\) satisfies the equation, so that it is a point on the plane.

Exercise 1: Use equation (1) to find the equation of the tangent planes to \( f(x, y) = 1 - x^2 - y^2 \) at the points (a) \((0, 0, 1)\) and (b) \((-2, 1, -4)\). Give a graphical explanation for your answer to (a).

An Alternative Formula for the Tangent Plane:

Equation (1) can be rewritten as
\[
f_x(x_0, y_0)x + f_y(x_0, y_0)y - z = d,
\]
(2)
where \( d \) is a constant chosen so that \((x_0, y_0, z_0)\) satisfies the equation. In other words, the tangent plane is the plane with normal vector \( \mathbf{n} = \langle f_x, f_y, -1 \rangle \) (evaluated at \((x_0, y_0)\)) passing through the point \((x_0, y_0, z_0)\). Formula (2) is a little easier to remember than equation (1). We will learn why the vector \( \mathbf{n} \) is truly perpendicular to the graph of the function in Section 11.6.
Linear Approximation to \( f(x, y) \) at \((x_0, y_0)\)

One of the key ideas in Calc 1 is that the tangent line is the best linear approximation to a function. The same result holds for functions of two or more variables: the tangent plane is the best linear approximation to a function. The linearization is obtained by solving equation (1) for \( z \) and recalling that \( z_0 = f(x_0, y_0) \). This gives

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

(3)

\( L(x, y) \) is called the linearization or linear approximation of \( f \) at \((x_0, y_0)\). Note that it is a linear function in the variables \( x \) and \( y \), that is, \( L \) is of the form \( L(x, y) = ax + by + c \) for some constants \( a, b, \) and \( c \).

Exercise 2: Find the linearization for \( f(x, y) = \sin(xy^2) + \sqrt{4x + y} \) about the point \((0, 1)\). Use it to estimate \( f(-0.1, 1.05) \). Compare your estimate with the actual function value.

Differentiability

Recall from Calc 1 that a function \( f(x) \) is differentiable at \( x_0 \) if \( f'(x_0) \) exists. The definition of differentiability is more complicated for functions of two or more variables, but intuitively, we say that \( z = f(x, y) \) is differentiable at \((x_0, y_0)\) if the linear approximation is a good approximation for points near \((x_0, y_0)\). In other words, differentiable functions are ones where the tangent plane approximates the function very well.

Example 1: Consider the functions \( f(x, y) = x^2 + y^2 \) and \( g(x, y) = \sqrt{x^2 + y^2} \) near the origin \((0, 0)\). Both functions have global minima at \((0, 0, 0)\) (see Figure 1). The tangent plane for \( f \) at \((0, 0)\) is

![Figure 1: The graph of \( f(x, y) = x^2 + y^2 \) and \( g(x, y) = \sqrt{x^2 + y^2} \) along with the plane \( z = 0 \).](image)
simply $z = 0$ because $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. As we zoom in on the graph of $f$ near the origin, it becomes flatter and flatter, and is well-approximated by its tangent plane. Thus $f$ is differentiable at the origin.

On the other hand, the first partial derivatives of $g$ do not exist at the origin. If we set $y = 0$, we have $g(x, 0) = \sqrt{x^2} = |x|$. Since $|x|$ is not differentiable at $x = 0$ (corner), $g_x(0, 0)$ does not exist. A similar argument applies to $g_y(0, 0)$. These facts are apparent in the graph of $g$ near the vertex of the cone. No matter how much we zoom into the graph of $g$, there will always be a cone point; consequently, $g$ is not well-approximated by the tangent plane $z = 0$, and is thus not differentiable at the origin.

The following fact is useful for determining whether a function is differentiable or not at a given point:

**Useful Fact:** If $f_x$ and $f_y$ exist and are continuous at $(x_0, y_0)$, then $f$ is differentiable at $(x_0, y_0)$.

**Exercise 3:** Consider the function

$$f(x, y) = \begin{cases} 
  \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\
  0 & \text{if } (x, y) = (0, 0). 
\end{cases}$$

Use the limit definition of the partial derivative to show that $f_x(0, 0) = f_y(0, 0) = 0$. Conclude that the tangent plane for $f$ at $(0, 0)$ is $z = 0$. Is the function differentiable at $(0, 0)$? Is it continuous at $(0, 0)$?