Multivariable Calculus, Spring 2005

Computer Project #4

Fun with Parameterizations: Cycloids and Lissajous Figures

DUE DATE: Friday, April 29th, in class.

1 Introduction

In this project you will learn about two very famous scientific curves: cycloids and Lissajous figures. The problems require the use of the software package MAPLE as well as some mathematics on your part to be done without the computer. The mathematical concepts needed for this project include trigonometry, parameterizing curves in the plane, velocity, acceleration, arclength and a small bit of number theory.

One of the goals of the project is for you to explore and learn about this aspect of multivariable calculus and to communicate your understanding and discoveries in a well-written and coherent report. It is **required** that you work in a group of two or three people, your original lab team. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced. Please turn in one report per group, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully and neatly, **writing in complete sentences.**

The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. It is also possible to write your answers within your MAPLE worksheet using the text option. Your presentation is important and I should be able to clearly read and understand what you are saying. Your report should provide answers to each of the questions below.

2 The Cycloid

The **cycloid** is a famous curve in mathematics obtained by rolling a circle along a flat surface and following the trajectory of a particular point on the circle. One of the interesting properties of a cycloid is that if a particle is moving in a constant force field, then the cycloid gives the path between two points which the particle travels over in the shortest amount of time. This is described as a **brachistochrone**.

Consider a wheel of radius 1 ft. which rests on the x-axis (the ground) and rolls without slipping along the positive x-axis. Suppose that at time $t = 0$, the center of the wheel is at $(0, 1)$ and that the point P we want to follow begins at the top of the wheel $(0, 2)$. Assume that the wheel is rolling at constant unit speed 1 ft/sec so that after t seconds, the center of the wheel is located at $(t, 1)$ and the wheel has rolled a distance of t feet.

(a) Show that the motion of the point P is given parametrically by the equations

$$\begin{aligned} x &= t + \sin t \\ y &= 1 + \cos t \end{aligned}$$

with $t \geq 0$.

(b) Use Maple to draw a plot of the cycloid over the time interval $[0, 4\pi]$. Do this by typing the commands

```
with(plots):
r := t -> (t + sin(t),1 + cos(t)):
plot([r(t),t=0..4*Pi]):
```

The second command assigns r to the vector with components $x(t)$ and $y(t)$. Print out your plot and label the points where the point P is on the ground, is at the same height as the center of the circle and is furthest from the ground.

(c) Find the times at which P reaches each of the 9 locations you identified in part **(b)**. Be sure to justify your answers.

(d) Give a parameterization for the wheel (not the cycloid) after it has rolled for s seconds. Make your parameterization go in the clockwise direction, starting from the top of the wheel.

(e) Using the result to part **(d)**, use Maple to plot four wheels and the cycloid on the same graph, drawing the wheel at times $s = 0, \pi/3, 2\pi/3, \pi$. Print out your plot and label the point P on each of the four wheels on your graph. To combine plots you can use the `display` command. For example, typing

```
cycl := plot([r(t),t=0..3*Pi],x=1..5):
line  := plot([t,t/(2*Pi),t=0..3*Pi],x=1..5):
display([cycl,line]);
```

will display the cycloid and a line on the same graph with the x range between 1 and 5.

(f) Find the velocity and speed of the cycloid. When is the cycloid going fastest and when is it going slowest? Interpret your findings with the physics of the motion. Sketch several velocity vectors on a graph of the cycloid paying particular attention to the length of the vectors.

(g) When is the velocity zero? Explain physically why this must happen.

(h) Notice that even though $x(t)$ and $y(t)$ are nice differentiable functions of t, the graph of the cycloid has a cusp (non-differentiability in the xy-plane). Compute the **direction** of the velocity vector as a function of time t. This unit vector $\vec{T}$ is usually called the **Unit Tangent Vector**. What happens to $\vec{T}$ just before and just after the cusp?

(i) Compute the arc length of the cycloid over one cycle of its orbit, that is, from $t = 0$ until the time the point P returns to a height of 2. Is the length greater than, less than or equal to the circumference of the wheel? Why does your answer make sense in physical terms? Note: To compute the integral, you either need to use the half-angle formulas from trigonometry or to rationalize the numerator. Please show your work.

3 Lissajous Figures

The French physicist Jules Antoine Lissajous (1822-1880) (pronounced Lee-suh-zhoo) was interested in studying waves and vibrations. Attaching mirrors to two tuning forks which were vibrating at *different* frequencies, he was able to project light off of the mirrors and onto one screen to create what is today called a **Lissajous figure**. His setup was similar to the modern device used to project laser light shows. Lissajous figures were used in the old days to determine the frequencies of sounds or radio signals. They found their way into popular culture in many sci-fi movies and TV shows, including the opening footage for *The Outer Limits* TV series. ("Do not attempt to adjust your picture — *we* are controlling the transmission.") [1, 2].

To create Lissajous figures, we will use the parameterization

$$\begin{aligned} x &= \cos\left(\frac{2\pi}{a}t\right) \\ y &= \sin\left(\frac{2\pi}{b}t\right) \end{aligned}$$

where a, b are always taken to be positive real numbers. The equations are set up so that the period of the horizontal component is a and the period of the vertical component is b. After loading the package `plots` in Maple, define the function $L(t)$ to be the x and y equations above. (See the cycloid problem for the correct syntax.) You will save a lot of time if you redefine a and b each time you wish to draw a Lissajous figure. For example, assuming you have defined L correctly, the commands

```
a := 3: b := 1:
plot([L(t),t=0..6]);
```

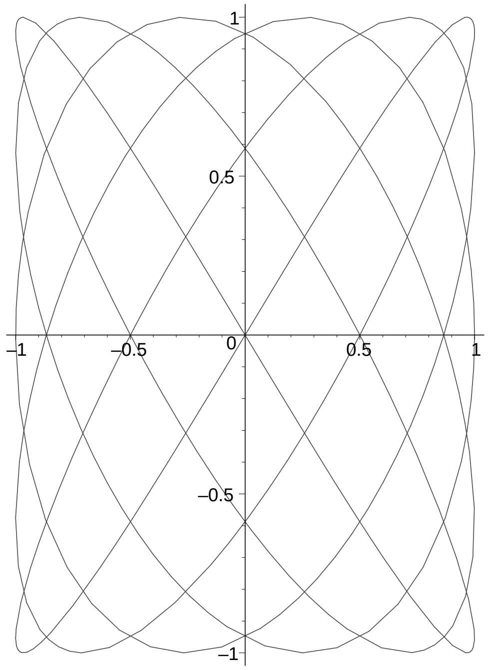
should draw the Lissajous figure with $a = 3$ and $b = 1$ over the time interval $[0, 6]$. Try plotting a few sample figures with different a and b values before answering the questions below.

(a) Begin by setting $a = b = 1$. What figure do you obtain? Using the parameterization above, what is the period of this figure? Explain. What figure will you obtain whenever $a = b$? No graphs need to be turned in for this part.

(b) Use Maple to plot the Lissajous figure for $a = 2$ and $b = 1$. Print out your plot and label at least 5 points on the figure with their corresponding times. What is the period of this curve? (ie. At what time are you back to where you began at $t = 0$ and will begin to retrace the same curve?)

(c) Keeping b fixed at 1 while varying a to $3, 4, 5, \ldots$, use Maple to plot the corresponding Lissajous curves. What changes do you see and why do they occur? What is the period when $b = 1$ and $a = 3$? What is the period when $b = 1$ and $a = k$, for any positive integer k?

(d) Use Maple to plot the Lissajous figure for $a = 1$ and $b = 2$. How is this different from the plot in part **(b)**? Explain. Would you say this curve is periodic? In other words, does it ever return to the starting point and then retrace its path again? If so, what is the period? What shape is this curve? Find the equation between x and y which describes this curve? Print out your plot and label at least 5 points on the figure with their corresponding times.



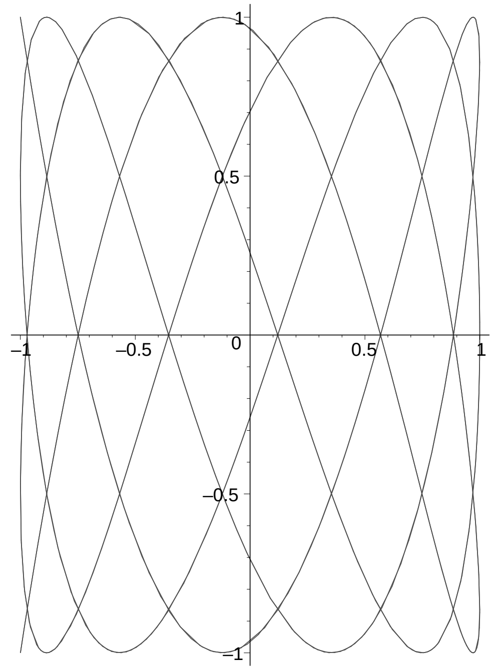


Figure 1: Two typical Lissajous figures.

(e) Keeping a fixed at 1 while varying b to $3, 4, 5, \ldots$, use Maple to plot the corresponding Lissajous curves. What changes do you see and why do they occur? What is the period when $a = 1$ and $b = 3$? What is the period when $a = 1$ and $b = k$, for any positive integer k? Do all the Lissajous curves obtained in this part close up on themselves (complete a cycle)? Which ones close up and which ones don't? Explain why.

(f) By experimenting with different integer values of a and b other than $a = 1, b = 1$, what can you conclude about the corresponding Lissajous figure? What role do a and b play? Try fixing one of a or b constant and varying the other to look for patterns in the figures. (Much of mathematical research begins with looking for patterns!) What is the period of the Lissajous figure if $a = 5$ and $b = 3$? What is the period of the Lissajous figure if $a = 4$ and $b = 6$? Find an expression for the period given any two integer values for a and b.

(g) Given the two Lissajous figures in Figure 1, find the integer values of a and b in each case. Explain how you obtained your answers. Note that the figure on the left closes up on itself while the figure on the right does not.

(h) All the examples above have used a and b as integers. What happens if you try $a = 1$ and $b = \sqrt{2}$? Is the curve periodic? What happens as you plot the curve over longer and longer time intervals? Explain.

References

[1] Hobbs Ed., *Lissajous Lab*, website: "http://www.math.com/students/wonders/lissajous/ lissajous.html" maintained by Math.com.

[2] MacTutor History of Mathematics archive, *Jules Antoine Lissajous*, website: "http://www-groups.dcs.st-andrews.ac.uk/history/Mathematicians/Lissajous.html" maintained by the School of Mathematics and Statistics, University of St. Andrews, Scotland.

[3] McCallum, W., Hughes-Hallett, D., et. al. Multivariable Calculus, John Wiley and Sons, Inc., New York, 1997.

[4] Wright, Eric, *Cycloids, Brachistochrones, and other Archaic Words*, Multivariable Calculus Computer Project, Dept. of Applied Math., University of Colorado, Boulder, Summer, 1999.