Multivariable Calculus, Spring 2015

Computer Project #3

Exploring the Bumpy Sphere using Spherical Coordinates

DUE DATE: Friday, May 1, 5:00 pm

1   Introduction

This project explores the properties of a family of functions known as *bumpy spheres* or sometimes referred to as *wrinkled spheres*. These spheres have been used to model tumors in medical research. Using Maple, you will graph these special spheres, compute their volume, and examine how these traits change for different parameter values. A common feature of mathematical research is investigating the properties of a family of functions (not just one function) as certain coefficients or powers vary. The natural coordinate system for visualizing and computing these spheres is, not surprisingly, spherical coordinates. The volume problems can actually be done by hand, but they take a long time to compute. Maple can do the integration much faster than we can.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced at the end of your report. The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Only **one project per group** should be submitted.

Your report should provide coherent answers to the questions in each section. Be sure to read carefully and answer all of the questions asked. Please do not overload your report (or my attention for reading) by including large numbers of graphs and tables. A well-written report with a few tables and graphs to illustrate key points is far better than a sloppy report with too many figures.

2   Bumpy Spheres

The family of functions we will be investigating for this project is given by the equation

$$\rho(\theta, \phi) \ = \ 1 + \frac{1}{5} \sin(a\,\theta) \sin(b\,\phi),$$

where a and b are always taken to be positive integers. In other words, this is a two-parameter family of functions that depends on the parameters a and b. Different choices of a and b lead to different graphs and volumes. Part of the goal of this project is to understand the effect that changing the parameters have on these features. Notice that ρ, the distance from the origin, depends on both the polar angle (θ) and the angle measured from the positive z-axis (ϕ).

To graph the function, you can use the `plot3d` command along with the option `coords = spherical`, in order to plot using spherical coordinates. For instance, try executing the following command:

```
plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical,scaling=constrained,axes=boxed)
```

If you want, you can choose from the Greek letters menu on the left to type θ instead of "theta," ϕ instead of "phi," and π instead of "Pi." Assuming the command runs correctly, you should obtain the unit sphere. This is because you have plotted the function $\rho = 1$ (the very first entry after the open parentheses in the `plot3d` command) over the intervals $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

Try executing the command

```
plot3d(1,theta=0..Pi,phi=Pi/2..Pi,coords=spherical,scaling=constrained,axes=boxed)
```

which is the same as the previous command except that now the angles range over the intervals $0 \le \theta \le \pi$ and $\pi/2 \le \phi \le \pi$. What is the resulting graph? Why does it look this way?

In order to graph our bumpy spheres, it helps to give the function a name, e.g., f, and define the parameters separately. The advantage of this approach is that you don't have to keep retyping the function every time you want to see its graph or compute the volume of the corresponding sphere. Use the following commands to define our family of functions and the parameters a and b.

```
f := (theta,phi) -> 1+(1/5)*sin(a*theta)*sin(b*phi):
a := 3:  b := 2:
```

Recall that a colon : suppresses the output of a Maple command. Once these commands have been executed, the function f and the parameter values a and b have been established. To see the effect of changing the parameter values, all you need to do is change the value assigned to a and b and then execute the `plot3d` command.

Try plotting $f(\theta, \phi)$ using $a = 3, b = 2$, and then again but with $a = 4, b = 1$. The results are shown in Figure 1. You will need to replace the 1 in the `plot3d` command above with $f(\theta, \phi)$. Some special options were added to create the plots in Figure 1. These are:

  `tickmarks=[5,5,5]`, `numpoints=1000`, `labels=[x,y,z]`

The `numpoints` command controls the number of points evaluated to produce the graph. The other two commands are straight forward.
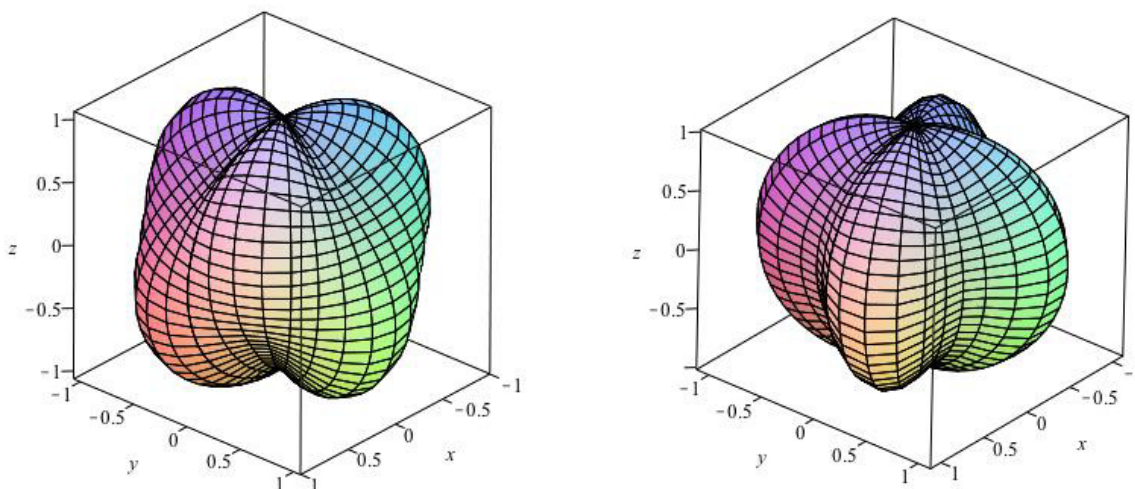

Figure 1: Two bumpy spheres. On the left is the function f with $a = 3$ and $b = 2$, while on the right is the graph of f with $a = 4$ and $b = 1$.

1. Begin by setting either a or b equal to zero. Use Maple to plot the graph of f. What figure do you obtain? Explain why.

2. Next set $a = 1$ and $b = 1$, and plot the resulting graph. What changes? Print out a plot that shows both this graph and the one from the previous question side by side. Find four points (in Cartesian coordinates (x, y, z)) that are common to each graph.

3. Keeping b fixed at 1, try varying a to $2, 3, 4, 5, \ldots$. Describe the changes you see in the graph as a increases. What role does a play in the resulting shape of the bumpy sphere? By examining the formula for f, explain mathematically why changing a has this effect. Print out one good representative bumpy sphere for this particular choice of parameter values.

4. Now keep a fixed at 1 and vary b to $2, 3, 4, 5, \ldots$. Describe the changes you see in the graph as b increases. What role does b play in the resulting shape of the bumpy sphere? By examining the formula for f, explain mathematically why changing b has this effect. Print out one good representative bumpy sphere for this particular choice of parameter values.

5. Suppose that the two bumpy spheres in Figure 2 are being used to model a pair of tumors. In each case, find the integer values of a and b used to generate the given graph. *Hint:* All of the parameter values used here are less than 10. The figure on the right was created with the setting `numpoints=4000`.
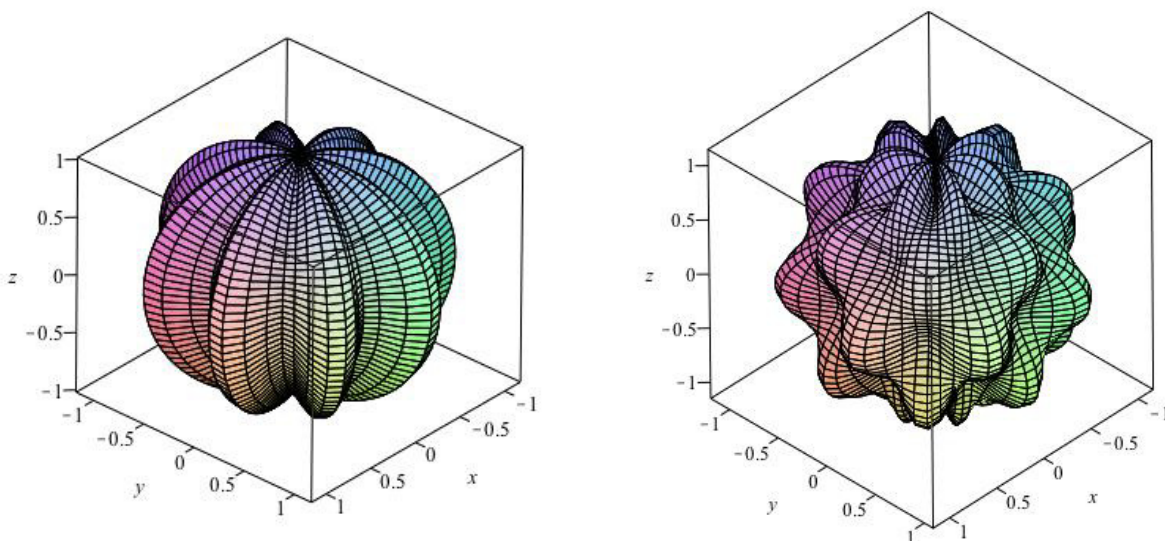

Figure 2: Two bumpy spheres used to model tumors. What are the corresponding values of a and b that generate each sphere?

3 The Volume of a Bumpy Sphere

In this section, we will compute the volume of different bumpy spheres, and describe the change in this volume as the parameters are varied. You will first need to set up a triple integral in spherical coordinates for the volume of a bumpy sphere. Use $d\rho \, d\theta \, d\phi$ as your order of integration. After doing the first integral by hand, you will arrive at a double integral that can be computed using Maple.

For example, to compute the double integral of $f(x,y) = x^2 y^9 - x^6 y^7$ over the region $-1 \leq x \leq 1$, $0 \leq y \leq 2$, we run the following commands in Maple. Try it.

```
with(Student[MultivariateCalculus]):
MultiInt(x^2*y^9-x^6*y^7, x =-1..1, y=0..2);
```

You should obtain $6208/105$ as the answer. If you wanted the value as a decimal, then you would place an `evalf( )` command around the `MultiInt` command, `evalf(MultiInt( ... ) )`. This gives 59.12380952 for the problem above.

To compute the volumes in this section, it helps to use the same set up as in the previous section, redefining the parameters a and b each time. **Remember:** every time you start Maple, you need to reload the package `with(Student[MultivariateCalculus]):` to get the `MultiInt` command to work.

1. What is the volume of the bumpy sphere when either $a = 0$ or $b = 0$? Why do we expect this answer?

2. What is the volume of the bumpy sphere when $a = 1$ and $b = 1$? Give the exact value as well as a numerical approximation rounded to five decimal places. Is this bumpy sphere bigger or smaller than the one from the previous question?

3. Keeping b fixed at 1, try varying a to $2, 3, 4, 5, \ldots$. What do you notice? What effect does changing a have on the volume of the corresponding bumpy sphere?

4. Now try keeping b fixed at 5, and varying a to $1, 2, 3, 4, 5, \ldots$. What do you notice? What effect does changing a have on the volume of the corresponding bumpy sphere? Based on this and the previous problem, (and perhaps a few more examples to be sure), make a bold mathematical statement about the effect of changing a on the volume of the bumpy sphere.

5. Now set $a = 1$, and vary b from 1 to 5. What happens this time? Are the corresponding spheres getting larger or smaller as b increases? Make a table showing the five different volumes (rounded to five decimal places) as b ranges from 1 to 5.

6. What happens to the volume of the bumpy sphere as $b \to \infty$? Give your answer accurate to eight decimal places. Bonus points if you can find the **exact** value (no decimals). Even more bonus points if you can **prove** that this is the exact value (i.e., do the integral by hand with a and b as parameters, etc.)