Multivariable Calculus, Spring 2015

Computer Project #1

Visualizing Functions of Two Variables

DUE DATE: Friday, March 13, start of class.

1 Introduction

This project explores ways of visualizing functions of two variables, $f(x, y)$, using the computer software program MAPLE. The two main graphical techniques are plotting graphs in three dimensions and drawing contour plots in two dimensions. The computer is a particularly useful device for understanding functions of more than one variable and you are encouraged to use the commands explained below throughout the course as needed. In this project you will search for a secret message, find the maximum and minimum heights of various functions, analyze contour plots in detail, and help an ant travel across a hot plate.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced at the end of your report. The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Only **one project per group** should be submitted.

Your report should provide coherent answers to the questions in Section 3. Be sure to read carefully and answer all of the questions asked. Please do not overload your report (or my attention for reading) by including large numbers of graphs and tables. A well-written report with a few tables and graphs to illustrate key points is far better than a sloppy report with too many figures.

Acknowledgments: Some of the ideas and equations for this project were borrowed from the lab project *Surfaces, contours, gradients and applications* by Kristian Sandberg, Department of Applied Mathematics, University of Colorado, Boulder, September, 1999 and from *Multivariable Calculus: Collaborative Learning Workbook* by David B. Damiano and Margaret N. Freije, Department of Mathematics and Computer Science, College of the Holy Cross, June, 2001.

2 Useful MAPLE Commands for Graphing

2.1 Drawing graphs in xyz-space

This project explores ways of visualizing functions of two variables, $f(x, y)$. One important representation of f is its **graph** in three dimensions. Letting z represent the output of the function, $z = f(x, y)$, we plot the set of points $(x, y, f(x, y))$ in xyz-space. You can think of the z-coordinate (the height) as representing the output of the function above the input variables (x, y). For instance, if the point $(2, 4, 5)$ is on the graph, then we know that $f(2, 4) = 5$. It would take a long, long, time to plot points in xyz-space by hand, but the computer is fast at piecing together data points to give a three-dimensional plot of the graph.

To utilize some of the commands helpful in visualizing graphs of two variables we need to load the MAPLE package `plots` by typing

```
with(plots);
```

You will see a list of commands that come with the package. To suppress this list you can use a colon rather than a semi-colon at the end of the command `with(plots):` . **Note:** You will need to run this command every time you start a new MAPLE session. To investigate one of the examples we have done in class (the bowl), type the following:

```
f := (x,y) -> x^2 + y^2:
plot3d(f(x,y),x=-3..3,y=-3..3,axes=boxed);
```

This should give you a graph of the function $f(x,y) = x^2 + y^2$. The `axes=boxed` command displays a coordinate box to give you a frame of reference. You could also try the `axes=framed` command which gives the axes without the outlining box. By clicking on your graph, you can adjust the frame of reference for your plot to get different views by dragging the mouse. This is very useful and really cool! There are also some buttons that you can click on at the top of the screen to see different features of the graph.

Draw a plot of $f(x,y) = x^2 - y^2$ and vary the frame of reference to get a good image of the standard saddle (i.e., a potato chip). Be sure to re-define $f(x,y)$ before you execute the `plot3d` command. To get more information and more options for the command `plot3d`, type `?plot3d`. To learn about any MAPLE command, type a question mark in front of it and hit return, or use the Help menu.

2.2 Contour Plots

The second method for interpreting a function $f(x,y)$ is the **contour plot**. The contour plot is a 2-d visualization of the function obtained by drawing the curves in the xy-plane corresponding to a *fixed* function value or height. These curves are called **level curves** or **contours**. (Think of a topographical map where a contour curve outlines the shape or contour of the land.) Here we fix the output value $z = c$ and sketch the curve in the xy-plane whose function values are all c, that is, graph the equation $f(x,y) = c$. This is equivalent to taking a cross-section $z = c$ in our 3-d graph and then projecting the image down onto the xy-plane. As we vary c uniformly (recall that contour values are always equally spaced), we obtain different contours, but we sketch them all on the same 2-d graph in the xy-plane. This yields the contour plot of the function.

To draw the contour plot of $f(x,y) = x^2 + y^2$, type:

```
contourplot(f(x,y), x=-3..3, y=-3..3);
```

Be sure to define the function f correctly. By default, MAPLE plots the contours for 8 different values of c (the function value) that are equally spaced. As we discussed in class, this does NOT mean that the contours themselves will be equally spaced! Light colored contours correspond to greater function values than darker colored ones. You can add more contours by inserting the command `contours = 15` or by specifying them precisely in a list such as

```
contours=[0,0.5,1,1.5,2,2.5,3,3.5,4]
```

In order to plot the contours, MAPLE samples the values of the function on a grid consisting of 25 points in the x-direction and 25 points in the y-direction. You can increase this amount (to get smoother contours) by typing `grid=[35,35]` for example. Try executing the following command:

```
contourplot(f(x,y),x=-3..3,y=-3..3,contours=[0,0.5,1,1.5,2,2.5,3,3.5,4],grid=[35,35]);
```

Note that the function $f(x,y) = x^2 + y^2$ has a **minimum** at the point $(0,0)$. This means that the function value $f(0,0) = 0$ is smaller than other function values nearby. This is easy to see from the 3d-plot since it is the point at the bottom of the bowl. What does the contour plot look like near the minimum? Would the picture be different for a maximum? Try plotting a contour map for $f(x,y) = -(x^2 + y^2)$ to see.

For the function $f(x,y) = x^2 - y^2$, the point $(0,0)$ is called a **saddle point**. (We will define this type of point later after learning about partial derivatives.) Intuitively, a saddle point has one direction in which the graph is concave up and another direction where the graph is concave down. So a saddle point is kind of a min/max all at once. What do the contours look like near the saddle point?

Two other useful commands for viewing contours are `densityplot` and `contourplot3d`. The first command yields a shaded plot of the domain of the function in the xy-plane where the shading corresponds to the function value. Lighter shades mean larger function values while darker shades correspond to smaller function values. Check out the density plot of the saddle by typing:

```
densityplot(x^2-y^2,x=-3..3,y=-3..3,grid=[50,50]);
```

The command `contourplot3d` is a version of the `plot3d` command that shows horizontal slices of the graph. It displays 15 slices by default. You can increase or decrease the number of slices by using the same options as with the `contourplot` command. Try the following to see the contours on the saddle in 3d:

```
contourplot3d(x^2-y^2,x=-3..3,y=-3..3,contours=20,axes=framed);
```

3 The Project

1. **A Secret Message:** Using `plot3d`, draw a 3-d plot of the function

$$f(x,y) = \cos(4x)\, e^{-(x^2+y^2/2)} \;+\; e^{-3((-x+0.5)^2+y^2/2)}$$

over the region $-3 < x < 3$, $-5 < y < 5$. This function can be assigned to f by typing

```
f := (x,y) -> cos(4*x)*exp(-(x^2+y^2/2)) + exp(-3*((-x+0.5)^2 + y^2/2);
```

To use this function in a command, you only need to type `f(x,y)` in place of where the expression would normally go. There is a secret message in the graph which can only be found if you restrict the output values (z-values) to be very small in magnitude. This can be accomplished using the option `view=0..5` which restricts the plot range for the z-coordinate between 0 and 5, for example. Type this command after the plot range for x and y are specified, but before you close the parentheses on the `plot3d` command.

   a. What is the secret message?

   b. What happens to the graph as x and y get very large in magnitude? Explain why mathematically by examining the formula for the function f.

Note: No graphs need to be turned in for this question.

2. Plot the function $f(x, y) = y^3 - 12y + 2x^2 + 4x + 4$ using `plot3d`.

   a. Using your graph and using the command `contourplot`, locate specifically any extrema (max's, min's or saddles). Give the coordinates $(x, y, f(x, y))$ for each extrema and label these points on a print out of a good 3d-graph. Later in the course, we will see how to find the extrema algebraically. Note that the command `evalf(f(-1,2));` can be used to find the numerical value of a function using MAPLE.

   b. Draw contour plots near each of the extrema. What do these contour plots have in common with those of the functions $h_1(x, y) = x^2 + y^2$ and $h_2(x, y) = x^2 - y^2$? How do you get the contour plot of the saddle to include the two crossing lines passing through the saddle point? Print out and turn in a good contour plot for each of the extrema you found in part **a.**

   c. Is there a global maximum or global minimum for this function? In other words, as (x, y) is varied over the whole xy-plane, is there a function value that is the largest or smallest possible? What would you say the range of this function is? Explain your answers.

3. Plot the function $g(x, y) = x^2/5 + 2e^{-x^2/2-y^2/2} - 2\sin x\, e^{-x^2/4-y^2/4}$ using `plot3d`. This function can be assigned to g by typing

```
g := (x,y) -> x^2/5 + 2*exp(-x^2/2 - y^2/2 ) -2*sin(x)*exp(-x^2/4 - y^2/4);
```

   a. Using your graph, the command `contourplot`, and any other commands you think helpful, locate as best as possible the extrema (there are a total of five). Give the coordinates $(x, y, g(x, y))$ for each point. No graphs are required to be turned in for this question.

   b. Is there any symmetry in this function which is demonstrated in the graphs? Where? How can this be seen analytically from the formula for the function? How does the symmetry help find the solution to part **a**?

   c. Is there a global maximum or a global minimum for $g(x, y)$? If so, what are they? What is the range of this function? Explain your answers.

4. **An Ant's Path:** Suppose that you are an ant who wants to cross a rectangular plate, $0 \le x \le 2\pi, -2 \le y \le 2$, whose temperature is given by the function $T(x, y) = (1 - y)\sin^2 x$. Starting on the side $y = -2$, your goal is to reach the side $y = 2$ by avoiding the hottest points on the plate.

   a. What are the coolest points on the side $y = -2$? Explain. Print out a contour plot of the temperature and draw a path that you would follow to get from one of the coolest points on $y = -2$ to one of the coolest points on $y = 2$. Recall that $\sin^2 x = (\sin x)^2$. **Note:** The number π is typed `Pi` in MAPLE, or it can be selected from the Greek menu on the left.

   b. What are the hottest points on the side $y = -2$? On the same contour plot as in part **a**, draw a path from one of the hottest points on the side $y = -2$ to one of the coolest point on the side $y = 2$.