MATH 241 Multivariable Calculus

Final Exam Review Questions

This is a **sample** list of the types of questions you can expect to find on the exam. Some of these questions were taken from past final exams.

1. Multiple Choice: Choose the best answer available.

   (a) The graph of the function $f(x, y) = \pi - x^2 + y^2$ is a

   (i) plane **(ii)** upside-down bowl **(iii)** saddle **(iv)** sphere **(v)** hyperboloid of one sheet

   (b) The directional derivative of $f(x, y, z) = xe^{xy/z}$ at the point $P(3, 0, 1)$ in the direction from P toward $Q(2, 2, 3)$ is

   (i) 17/3 **(ii)** 6 **(iii)** 7 **(iv)** 22/3 **(v)** 23/3

   (c) Suppose that $f(x, y) = 3x^2y - 2y^3 + 4$. The direction of greatest increase for f at the point $(-1, 1)$ is

   (i) $-3\vec{i} + 6\vec{j}$ **(ii)** $\frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}$

   (iii) $-\frac{2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}$ **(iv)** $-\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

   (d) Suppose that $z = f(x, y)$ is a differentiable function of x and y, and that $x(u, v) = u - v$ and $y(u, v) = u$. Then $z_u + z_v =$

   (i) 0 **(ii)** f_x **(iii)** f_y **(iv)** $f_x + f_y$

   (e) The integral $\int_{-1}^{1}\int_{0}^{1-x^2} f(x, y)\, dy\, dx$ is equivalent to

   (i) $\int_{-1}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y)\, dx\, dy$ **(ii)** $\int_{-1}^{1}\int_{0}^{\sqrt{1-y}} f(x, y)\, dx\, dy$

   (iii) $\int_{0}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y)\, dx\, dy$ **(iv)** $\int_{0}^{1}\int_{0}^{\sqrt{1-y}} f(x, y)\, dx\, dy$

2. Given points $A(3, 0, -1), B = (4, -2, 1)$ and $C(5, 0, 1)$:

   (a) Find the vectors $\vec{AB}, \vec{AC}$ and $\vec{BC}$.

   (b) Which two of the three points are closest to each other? Explain.

   (c) Find the angle between the vectors $\vec{AB}$ and $\vec{AC}$.

   (d) Find parametric equations for the line passing through C and perpendicular to the plane containing A, B, and C.

3. Find the unit tangent and normal vectors to the curve $\mathbf{r}(t) =< t, 2t, t^2 >$ when $t = 1$.

4. Harry Potter and Draco Malfoy are each seekers on their respective Quidditch teams searching for the golden switch. While flying on their brooms, both lads collide at the point $(1, 0, 0)$ and notice the snitch hovering overhead at the point $(1, 0, 6\pi)$ (assume the units are in meters). Crabbe and Goyle, Malfoy's mischievous malcontents, have placed a curse on Harry's broom, forcing him to fly along the standard path of a helix to get to the snitch, while Malfloy gets to fly in a straight line directly toward the snitch. However, Ron and Hermione have cursed Malfoy's broom so that he can only fly at 2/3 of the speed of Harry.

   (a) What is the standard parametrization of the helix? How many loops does Harry have to make before he reaches the snitch?

   (b) What speed is Harry flying at and how long does it take him to reach the snitch?

   (c) How far does Malfoy have to fly to reach the snitch?

   (d) When does Malfoy reach the snitch? Who wins the match, Gryffindor or Slytherin?

5. Let $g(x, y) = \sin(xy) + 4$.

   (a) State the domain and range of g.

   (b) Show that $(0, 0)$ is a critical point for g and use the second derivative test to classify it as either a maximum, minimum, or saddle point.

   (c) Describe the level curves near the point $(0, 0)$. Give a rough sketch of the contour plot near $(0, 0)$.

6. Let $f(x, y) = -x^4 + 4xy - 2y^2 - 1$.

   (a) Find all the critical points of $f(x, y)$.

   (b) Use the second-derivative test to classify each critical point as either a maximum, minimum or saddle point.

   (c) Does $f(x, y)$ have an absolute max or min? Explain.

7. Find the greatest product three numbers can have if the sum of their squares must be 48.

8. **(a)** By reversing the order of integration, find the value of

$$\int_0^1 \int_{\sqrt{x}}^1 \cos(\pi y^3)\ dy\, dx\,.$$

   (b) Find the volume of the ice-cream cone bounded below by $z = \sqrt{3(x^2 + y^2)}$ and bounded above by $x^2 + y^2 + z^2 = 8$.

9. Consider the two-dimensional vector field

$$\mathbf{F} = (ye^{xy} + \cos(x-y))\,\mathbf{i} + (xe^{xy} - \cos(x-y) + 3)\,\mathbf{j}$$

(a) Show that the curl $\mathbf{F} = 0$.

(b) Find a potential function $f(x,y)$ for $\mathbf{F}$, that is, find a function f such that $\nabla f = \mathbf{F}$.

(c) Calculate the line integral for $\mathbf{F}$ along the path C which starts at $(2\pi, 0)$, heads along a line segment to the point $(5\pi, \ln\pi)$ and then heads along another line segment to the point $(0, -\pi/2)$.

10. Use Green's Theorem to find the value of the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = x^2y^2\,\mathbf{i} - 3e^{y^2}\,\mathbf{j}$ and C is the circle $x^2 + y^2 = 5$ traversed counterclockwise.

11. In this final problem you will derive one of the great formulas of mathematics:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}. \tag{1}$$

This integral has important applications in probability and statistics. Notice that it is impossible to do the integral analytically using Calc 1 or 2 methods. The trick to doing the integral is to go UP one dimension!

(a) Consider the double integral

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

Describe the region of integration in polar coordinates. Change the integral into polar coordinates and evaluate. *Note:* You may use the slightly crude notation $e^{-\infty} = 0$ if you wish.

(b) Explain why

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \left(\int_0^\infty e^{-x^2}\,dx\right)^2.$$

Hint: Do a bit of manipulation and then recall the concept of a "dummy" variable.

(c) Use the results from parts **(a)** and **(b)** to derive formula (1).