MATH 241 Multivariable Calculus

Exam #3 SOLUTIONS April 22, 2015 Prof. G. Roberts

1. Multiple Choice: Choose the **best** answer available (no work required). (5 pts. each)

(a) Suppose that $(3, 2)$ is a critical point of the function $f(x, y)$, and that all the partial derivatives of f exist and are continuous. If $f_{xx}(3, 2) = -4$ and $f_{yy}(3, 2) = 1$, then, at the point $(3, 2)$, the function $f(x, y)$ has

(i) a maximum,
(ii) a minimum,
(iii) a saddle point,
(iv) not enough information given.

Answer: (iii). Applying the second derivative test, the determinant of the Hessian matrix at the critical point $(3, 2)$ is equal to

$$D \;=\; f_{xx}(3,2)\cdot f_{yy}(3,2) - (f_{xy}(3,2))^2 \;=\; -4 - (f_{xy}(3,2))^2 \;<\; 0.$$

Thus, $D < 0$ regardless of the value of $f_{xy}(3, 2)$, and so $(3, 2)$ is a saddle point.

Alternatively, since $f_{xx}(3, 2) > 0$, the function is concave up in the x-direction, but since $f_{yy}(3, 2) < 0$, it is concave down in the y-direction. This is enough qualitative information to conclude that the critical point is a saddle.

(b) Let R be the rectangle $[-1, 1] \times [0, 4]$. Which of the following is the Riemann sum for $\int\int_R g(x, y)\, dA$ with $m = n = 2$, assuming that the sample points are chosen to be the upper right corners?

(i) $[g(-1, 0) + g(1, 0) + g(-1, 4) + g(1, 4)] \cdot 8$,
(ii) $[g(0, 0) + g(1, 0) + g(0, 4) + g(1, 4)] \cdot 8$,
(iii) $[g(0, 2) + g(1, 2) + g(1, 0) + g(0, 0)] \cdot 2$,
(iv) $[g(0, 2) + g(1, 2) + g(1, 4) + g(0, 4)] \cdot 2$.

Answer: (iv). To compute the Riemann sum, we make two subdivisions in both the x- and y-directions. Hence, $\Delta x = (1 - -1)/2 = 1$ and $\Delta y = (4 - 0)/2 = 2$, so $\Delta A = 1 \cdot 2 = 2$ (this eliminates the first two choices). Dividing the rectangle into four sub-rectangles and choosing the upper right corners of each sub-rectangle as the sample points leads to choice **(iv)**.

(c) Suppose that D is a region in the plane whose area is $\sqrt{2}$. The value of the double integral $\int\int_D 5\, dA$ is

(i) 25,
(ii) $5\sqrt{2}$,
(iii) $\sqrt{2}$,
(iv) not enough information given.

Answer: **(ii)**. Recall that the double integral of 1 over the region R is equal to the area of R. Therefore, we have $\int\int_D 5\ dA\ = 5\int\int_D 1\ dA\ = 5\sqrt{2}$.

(d) In cylindrical coordinates, assuming that $r \geq 0$, the equation $z + r = 0$ describes a

(i) half plane,

(ii) cylinder,

(iii) sphere,

(iv) upward facing cone,

(v) downward facing cone.

Answer: **(v)**. In cylindrical coordinates, $r = \sqrt{x^2 + y^2}$, so that $z = -r$ is equivalent to $z = -\sqrt{x^2 + y^2}$, a downward facing cone with vertex at the origin.

2. Using the method of Lagrange multipliers, find the absolute maximum and minimum values of $f(x, y) = 2x + y^2$ subject to the constraint $g(x, y) = 2x^2 + y^2 = 8$.

State both the maximum and minimum values as well as the point(s) where each occurs. (20 pts.)

Answer: The minimum value of f is -4, which occurs at the point $(-2, 0)$. The maximum value is $17/2$, occurring at the points $(1/2, \pm\sqrt{15/2})$.

Using Lagrange multipliers, we must solve the system of equations $\nabla f = \lambda\nabla g$ and $g = 8$. This leads to the system

$$\begin{aligned} 2 &= \lambda 4x & (1) \\ 2y &= \lambda 2y & (2) \\ 2x^2 + y^2 &= 8. & (3) \end{aligned}$$

The easiest equation to handle is equation (2). We have $2y - \lambda 2y = 0$ or $2y(1 - \lambda) = 0$. Thus there are two cases: $y = 0$ or $\lambda = 1$.

Case 1: Suppose that $y = 0$. Plugging into the constraint given by equation (3), we see that $2x^2 = 8$ or $x = \pm 2$. (It follows from equation (1) that $\lambda = \pm 1/4$, but this is irrelevant.) Thus, two of our possible extrema occur at $(\pm 2, 0)$.

Case 2: Suppose that $\lambda = 1$. Plugging into equation (1), we find that $2 = 4x$ or $x = 1/2$. Plugging this value into equation (3) gives $1/2 + y^2 = 8$ or $y = \pm\sqrt{15/2}$. Thus, there are two more possible extrema at the points $(1/2, \pm\sqrt{15/2})$.

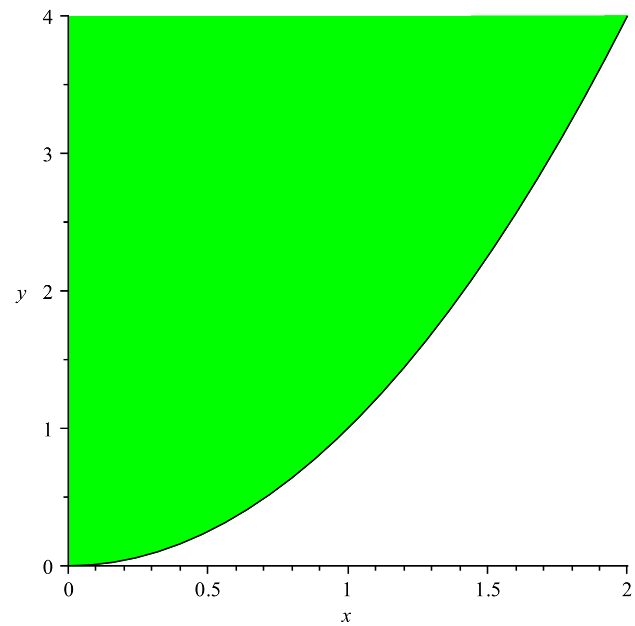
To finish the problem, we evaluate the function f at each of the four points just determined. We have $f(2, 0) = 4$, $f(-2, 0) = -4$, and $f(1/2, \pm\sqrt{15/2}) = 17/2$. Hence, the absolute maximum is $17/2$ at the points $(1/2, \pm\sqrt{15/2})$ and the absolute minimum is -4 at the point $(-2, 0)$.

3. Consider the integral

$$\int_0^2 \int_{x^2}^4 2xe^{y^2}\, dy\, dx\,.$$

(a) Sketch the region of integration. (6 pts.)

Answer: The region of integration is defined by the inequalities $x^2 \leq y \leq 4$ and $0 \leq x \leq 2$ (vertical cross-sections). This gives the region in green shown below.



(b) Reverse the order of integration and evaluate the integral. (14 pts.)

Answer: $\frac{1}{2}(e^{16}-1)$.

Note that the integral is impossible to compute in its current form. To reverse the order of integration, we switch from vertical to horizontal cross-sections. Note that $y = x^2$ is equivalent to $x = \sqrt{y}$. Thus, an alternative way to describe the region of integration is $0 \leq x \leq \sqrt{y}$ and $0 \leq y \leq 4$.

We have

$$\begin{aligned}
\int_0^2 \int_{x^2}^4 2xe^{y^2}\, dy\, dx &= \int_0^4 \int_0^{\sqrt{y}} 2xe^{y^2}\, dx\, dy \\
&= \int_0^4 x^2 e^{y^2}\Big|_0^{\sqrt{y}}\, dy \\
&= \int_0^4 ye^{y^2}\, dy \quad \text{(a u-sub with $u = y^2$, $du = 2y\, dy$)} \\
&= \frac{1}{2}e^{y^2}\Big|_0^4 \\
&= \frac{1}{2}\left(e^{16}-1\right).
\end{aligned}$$

4. Using polar coordinates, evaluate

$$\iint_D x \, dA\,,$$

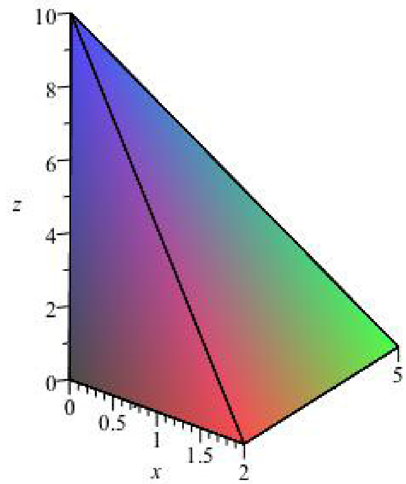
where D is the region lying to the right of the y-axis (so $x \geq 0$) and between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. (14 pts.)

Answer: 14/3. The radii of the two circles are $r = 1$ and $r = 2$, respectively. Thus, in polar coordinates, the region of integration is $1 \leq r \leq 2$, $-\pi/2 \leq \theta \leq \pi/2$. Since $x = r\cos\theta$ in polar coordinates, we have

$$\begin{aligned}
\iint_D x \, dA &= \int_{-\pi/2}^{\pi/2} \int_1^2 r\cos\theta \cdot r \, dr \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_1^2 r^2 \cos\theta \, dr \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left. \frac{r^3}{3} \cos\theta \right|_1^2 d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{7}{3} \cos\theta \, d\theta \\
&= \left. \frac{7}{3} \sin\theta \right|_{-\pi/2}^{\pi/2} \\
&= \frac{7}{3}\,(1 - -1) \;=\; \frac{14}{3}.
\end{aligned}$$

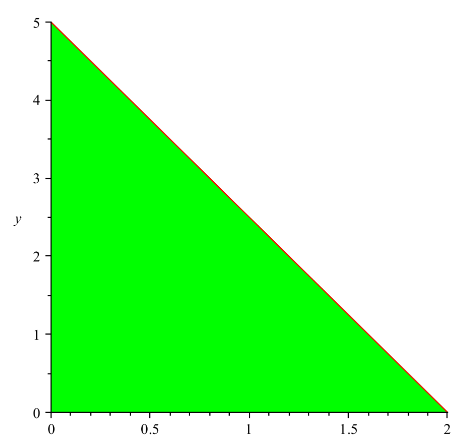
5. Set up a triple integral to find the volume of the tetrahedron enclosed by the three coordinate planes and the plane $5x + 2y + z = 10$. **DO NOT** evaluate the triple integral, just **set it up**. (10 pts.)

Answer: It is helpful to start by finding the x-, y- and z-intercepts of the plane $5x+2y+z = 10$. For example, when $x = y = 0$, we have $z = 10$. The x-intercept is 2 and the y-intercept is 5. The tetrahedron is shown below.



Using vertical line segments parallel to the z-axis, we see that $0 \leq z \leq 10 - 5x - 2y$ (starting in the xy-plane and exiting in the plane $z = 10 - 5x - 2y$.) Then we project the entire solid into the xy-plane (see figure below). The plane $z = 10 - 5x - 2y$ intersects the xy-plane in the line $y = 5 - \frac{5}{2}x$. This can be found by plugging in $z = 0$ into the equation for the plane, or by using the intercepts to find the equation of the line. Using vertical cross-sections to describe the projection into the xy-plane, we have $0 \leq y \leq 5 - \frac{5}{2}x$ and $0 \leq x \leq 2$. The triple integral for the volume is therefore

$$V = \int_0^2 \int_0^{5-\frac{5}{2}x} \int_0^{10-5x-2y} 1 \, dz \, dy \, dx \, .$$



6. Evaluate

$$\iiint_S (x^2+y^2+z^2)^{3/2}\, dV\,,$$

where S is the solid that lies within the sphere $x^2+y^2+z^2=2$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$. (16 pts.)

Answer: $\frac{4\sqrt{2}\,\pi}{3}$. Using spherical coordinates, the sphere $x^2+y^2+z^2=2$ is simply $\rho=\sqrt{2}$, and the cone $z=\sqrt{x^2+y^2}$ is just $\phi=\pi/4$. Since the cone is the upper-limit and the xy-plane is the lower limit, we have $\pi/4 \le \phi \le \pi/2$ (remember that ϕ is the angle measured from the positive z-axis). The other limits of integration are $0\le\theta\le 2\pi$ and $0\le\rho\le\sqrt{2}$. The integrand $(x^2+y^2+z^2)^{3/2}$ simplifies to $(\rho^2)^{3/2}=\rho^3$. We compute

$$\begin{aligned}
\iiint_S (x^2+y^2+z^2)^{3/2}\, dV &= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \rho^3\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \rho^5\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \left.\frac{\rho^6}{6}\right|_0^{\sqrt{2}} \sin\phi\, d\phi\, d\theta \\
&= \frac{8}{6}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \sin\phi\, d\phi\, d\theta \\
&= \frac{4}{3}\int_0^{2\pi} \left.-\cos\phi\right|_{\pi/4}^{\pi/2} d\theta \\
&= \frac{4}{3}\cdot\frac{\sqrt{2}}{2}\int_0^{2\pi} 1\, d\theta \\
&= \frac{2\sqrt{2}}{3}\cdot 2\pi = \frac{4\sqrt{2}\,\pi}{3}.
\end{aligned}$$