MATH 241 Multivariable Calculus

Exam #2 SOLUTIONS **March 25, 2015** **Prof. G. Roberts**

1. Multiple Choice: Choose the **best** answer available (no work required). (5 pts. each)

(a) Suppose that the curvature $\kappa(t)$ of the path of a moving particle is always zero. Which of the following is **NOT** necessarily true?

(i) The particle is traveling along a straight line.

(ii) The normal component of the acceleration a_N is zero.

(iii) The unit tangent vector $\mathbf{T}(t)$ is constant.

(iv) The speed of the particle is constant.

Answer: (iv). Since the curvature is always zero, the path traced out by the particle must be a straight line. The formula $a_N = \kappa|\mathbf{r}'(t)|^2$ shows that $a_N = 0$ and the formula $\kappa = |d\mathbf{T}/ds|$ implies that the unit tangent vector is constant. (Note that this also follows because the path of the particle is on a straight line.) The speed $|\mathbf{r}'(t)|$, however, does not have to be constant (the particle could be accelerating or decelerating along its line of motion).

(b) Evaluate the following limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{3x^4 + y^4}$

(i) 0

(ii) 1/4

(iii) 1/3

(iv) The limit does not exist.

Answer: (iv). The limit does not exist. If we choose a path along the x-axis (setting $y = 0$), we find $\displaystyle\lim_{x\to 0} \frac{0}{3x^4 + 0} = 0$, but taking a path along the line $y = x$, we obtain $\displaystyle\lim_{x\to 0} \frac{x^4}{3x^4 + x^4} = 1/4$. Since these two values are not equal, the limit does not exist.

(c) If it exists, the value of $\displaystyle\lim_{h\to 0} \frac{f(-1, 3+h) - f(-1, 3)}{h}$ equals

(i) $f_x(-1, 3)$

(ii) $f_y(-1, 3)$

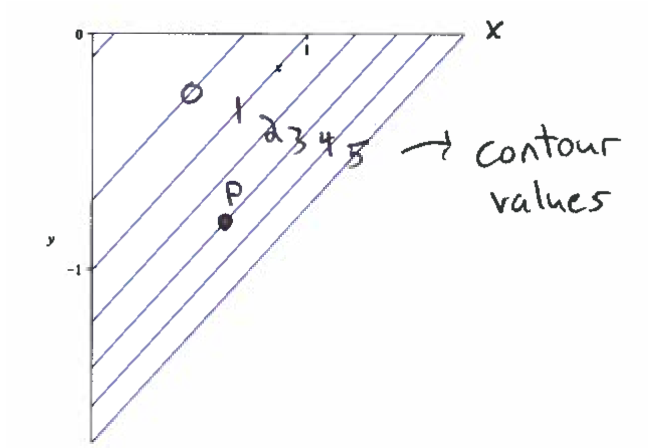
(iii) $f_{xy}(-1, 3)$

(iv) $D_{-\mathbf{i}}\, f(-1, 3)$

(v) $D_{3\mathbf{j}}\, f(-1, 3)$

Answer: (ii). This is the limit definition for the partial derivative of f with respect to y at the point $(-1, 3)$.

(d) Consider the contour plot of the function $f(x, y)$ shown below. Which of the following best describes the signs of the partial derivatives at the point P?



(i) $f_x > 0,\ f_y > 0,\ f_{yy} > 0$
(ii) $f_x > 0,\ f_y < 0,\ f_{yy} < 0$
(iii) $f_x > 0,\ f_y < 0,\ f_{yy} > 0$
(iv) $f_x < 0,\ f_y > 0,\ f_{yy} > 0$
(v) $f_x < 0,\ f_y < 0,\ f_{yy} < 0$

Answer: (iii). Starting at the point P and moving in the positive x-direction, we see that the function values are increasing, so $f_x > 0$. However, moving in the positive y-direction, the contour values decrease, so $f_y < 0$. Since the contours are spreading apart in the positive y-direction, the rate of decrease in the y-direction is increasing (e.g., the slopes might be something like -4, then -2, then -1). Thus, the values of f_y are increasing in the positive y-direction and consequently, $f_{yy} > 0$.

2. Consider the parametrized curve $\mathbf{r}(t) = 3t^2\,\mathbf{i} + (2 - t^2)\,\mathbf{j} + (-1 + \frac{3}{2}t^2)\,\mathbf{k}$, with $t \geq 0$. (19 pts.)

(a) Find the speed at which the curve is being traced out.

Answer: We have $\mathbf{r}'(t) =< 6t, -2t, 3t >$, so that

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{(6t)^2 + (-2t)^2 + (3t)^2} \\ &= \sqrt{36t^2 + 4t^2 + 9t^2} \\ &= \sqrt{49t^2} \\ &= 7t \quad \text{since } t \geq 0. \end{aligned}$$

(b) Reparametrize the curve with respect to arc length s measured from the point where $t = 0$.

Answer: Using the result from part **(a)**, we must solve the differential equation

$$\frac{ds}{dt} = 7t, \qquad s(0) = 0.$$

Separating and integrating leads to $ds = 7t\,dt$, and then $s(t) = \frac{7}{2}t^2 + c$. Since $s = 0$ when $t = 0$, we quickly find that $c = 0$. Therefore $s = \frac{7}{2}t^2$, or, after solving for t, $t = \sqrt{2s/7}$.

Substituting the value for t back into the original parametrization yields

$$\begin{aligned}\mathbf{r}(t(s)) &= 3\left(\frac{2s}{7}\right)\mathbf{i} + \left(2 - \frac{2s}{7}\right)\mathbf{j} + \left(-1 + \frac{3}{2}\cdot\frac{2s}{7}\right)\mathbf{k} \\ &= \frac{6s}{7}\mathbf{i} + \left(2 - \frac{2s}{7}\right)\mathbf{j} + \left(-1 + \frac{3s}{7}\right)\mathbf{k}.\end{aligned}$$

(c) What is the curve being traced out? Be as specific as possible.

Answer: The curve being traced out is a line since it can be written in the standard form $\mathbf{r}(s) = \mathbf{P} + \mathbf{v}s$. The line starts at the point $P = (0, 2, -1)$ and heads in the direction of the vector $\mathbf{v} =< 6/7, -2/7, 3/7 >$.

3. Consider the function $z = g(x, y) = \ln(\sqrt{x^2 + y^2}\,)$. (15 pts.)

(a) State the domain and range of $g(x, y)$.

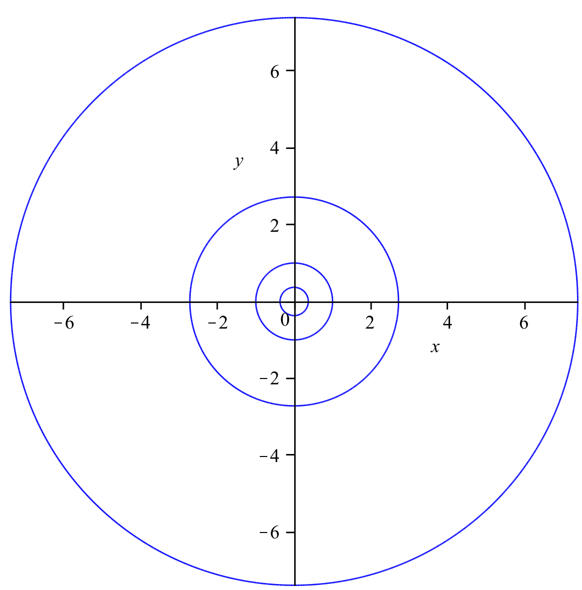
Answer: The domain is all points in the xy-plane except for the origin $(0, 0)$. This can be written as $\mathbb{R}^2 - \{(0, 0)\}$. We must exclude the origin because $\ln(0)$ is undefined. The range of g is all real numbers, or $(-\infty, \infty)$, because that is the range of the natural log function. (All positive real values will be plugged into the log function because $\sqrt{x^2 + y^2}$ can output any positive real number.)

(b) Sketch the level curves (contours) of $g(x, y)$ using the values $c = -1, 0, 1, 2$.

Answer: The level curves are found by solving $g(x, y) = c$. This gives $\ln(\sqrt{x^2 + y^2}) = c$ or $\sqrt{x^2 + y^2} = e^c$. Squaring both sides of this last equation, we obtain

$$x^2 + y^2 = (e^c)^2,$$

which means that the level curves are circles centered at $(0, 0)$ with radius $r = e^c$. For the values $c = -1, 0, 1, 2$, the radii of the four contours are approximately $0.37, 1, 2.7, 7.4$, respectively. The contour plot is shown below.



4. Find the equation of the tangent plane to the graph of the function (a surface)

$$z = f(x, y) = y\sin(x - y) - 5$$

at the point $(3, 3, -5)$. (10 pts.)

Answer: $z = 3x - 3y - 5$. We have $f_x = y\cos(x - y) \cdot 1$ so that $f_x(3, 3) = 3 \cdot \cos(0) = 3$, and $f_y(x, y) = \sin(x - y) + y \cdot \cos(x - y) \cdot (-1)$ (product rule and chain rule). Thus, $f_y(3, 3) = 0 + 3 \cdot \cos(0) \cdot (-1) = -3$.

Using the formula for the equation of a tangent plane to the graph of a function,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

we have

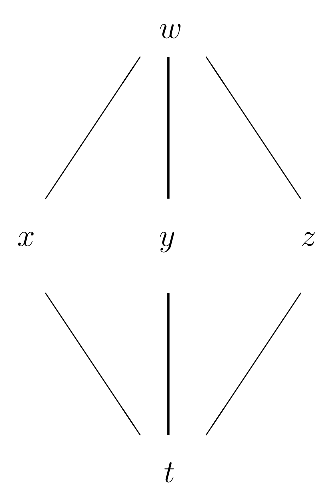
$$z + 5 = 3(x - 3) + -3(y - 3),$$

which simplifies to $z = 3x - 3y - 5$.

5. Suppose that $w = f(x, y, z)$ is a differentiable function of three variables, and that $x = x(t)$, $y = y(t)$, and $z = z(t)$ are each differentiable functions of one variable. (16 pts.)

(a) Use a tree diagram to write out the Chain Rule for dw/dt.

Answer: The tree diagram for dw/dt is shown below.



From the diagram, we have that

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}.$$

(b) Suppose that $w = f(x, y, z) = x^2 y e^{x-z}$ and that

$$x(2) = 3 \qquad y(2) = -1 \qquad z(2) = 3$$
$$x'(2) = 1/3 \qquad y'(2) = 0 \qquad z'(2) = 4.$$

Find dw/dt when $t = 2$.

Answer: 31. First we compute the partials of f. We find that

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2xye^{x-z} + x^2ye^{x-z}\cdot 1 \;=\; xye^{x-z}(2+x)\\ \frac{\partial f}{\partial y} &= x^2e^{x-z}\\ \frac{\partial f}{\partial z} &= x^2ye^{x-z}\cdot(-1) \;=\; -x^2ye^{x-z}.\end{aligned}$$

Next, since $t = 2$, we plug in the values of x, y and z at $t = 2$ into each partial. These values are given as $x(2) = 3, y(2) = -1$ and $z(2) = 3$. In other words, $(x, y, z) = (3, -1, 3)$. We compute that $f_x(3, -1, 3) = -15$, $f_y(3, -1, 3) = 9$, and $f_z(3, -1, 3) = 9$.
Using our chain rule from part **(a)**, we find that

$$\begin{aligned}\frac{dw}{dt} &= -15\cdot\frac{1}{3} \;+\; 9\cdot 0 \;+\; 9\cdot 4\\ &= -5+36 \;=\; 31.\end{aligned}$$

6. Let $f(x, y) = \sqrt{x^2 + 2y^2}$. (20 pts.)

(a) Find and simplify the gradient of $f(x, y)$.

Answer: We compute that

$$\begin{aligned}f_x &= \frac{1}{2}(x^2+2y^2)^{-1/2}\cdot 2x \;=\; \frac{x}{\sqrt{x^2+2y^2}}\\ f_y &= \frac{1}{2}(x^2+2y^2)^{-1/2}\cdot 4y \;=\; \frac{2y}{\sqrt{x^2+2y^2}}.\end{aligned}$$

Therefore,

$$\nabla f \;= < \frac{x}{\sqrt{x^2+2y^2}}, \frac{2y}{\sqrt{x^2+2y^2}} > .$$

(b) Find the directional derivative of $f(x, y)$ at the point $(-1, 2)$ in the direction of the vector $\mathbf{v} = -3\,\mathbf{i} + 4\,\mathbf{j}$.

Answer: 19/15. We use the formula $D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u}$ for the directional derivative. First, we must convert $\mathbf{v}$ into a unit vector by dividing by its length, $|\mathbf{v}| = 5$. This gives

$$\mathbf{u} \;=\; \frac{1}{5} < -3, 4 > = < -\frac{3}{5}, \frac{4}{5} > .$$

Then, using the formula from part **(a)**, we compute that

$$\nabla f(-1, 2) \;= < -\frac{1}{3}, \frac{4}{3} > .$$

Taking the dot product of these two vectors, we find $D_{\mathbf{u}}f(-1, 2) = 3/15 + 16/15 = 19/15$.

(c) Find the direction (as a unit vector) of greatest **decrease** for f at the point $(-1, 2)$.

Answer: Recall that ∇f points in the direction of greatest increase. Therefore, $-\nabla f$ points in the direction of greatest decrease. From part **(b)**, we see that

$$-\nabla f(-1,2) \;=< \frac{1}{3}, -\frac{4}{3} > .$$

To make this into a unit vector, we divide by its length,

$$|-\nabla f(-1,2)| \;=\; \sqrt{\frac{1}{9}+\frac{16}{9}} \;=\; \frac{\sqrt{17}}{3}.$$

Our final answer is therefore

$$\frac{3}{\sqrt{17}} < \frac{1}{3}, -\frac{4}{3} > \quad \text{or} \quad < \frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}} > .$$