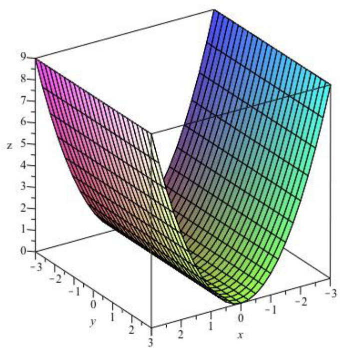
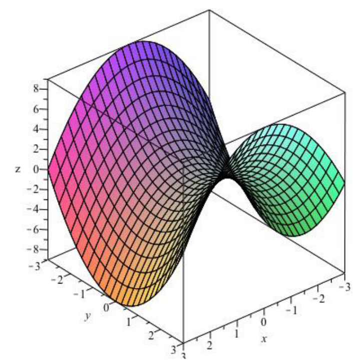
MATH 241 Multivariable Calculus

Exam #1 SOLUTIONS **February 20, 2015** **Prof. G. Roberts**

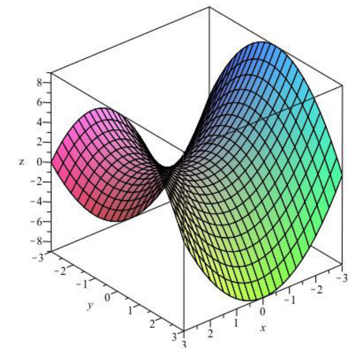
1. Match the graph with the correct function. There is exactly **one** function for each graph. Provide a brief explanation for your choices. (18 pts.)



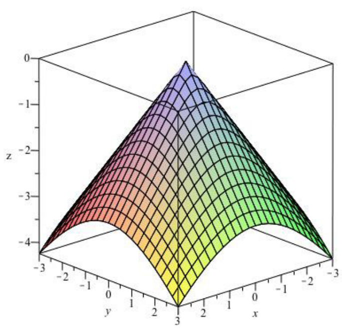
(a)



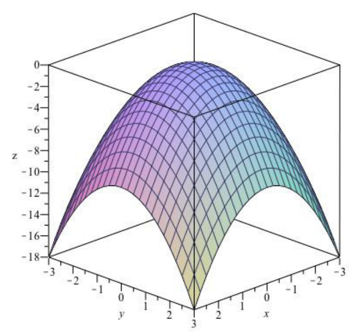
(b)



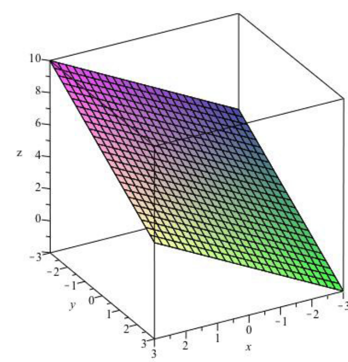
(c)



(d)



(e)



(f)

(i) $f(x,y) = x^2 - y^2$ **(ii)** $f(x,y) = y^2 - x^2$ **(iii)** $f(x,y) = -x^2 - y^2$

(iv) $f(x,y) = 4 + x - y$ **(v)** $f(x,y) = x^2$ **(vi)** $f(x,y) = -\sqrt{x^2 + y^2}$

(a) __(v)__ **(b)** __(ii)__ **(c)** __(i)__

(d) __(vi)__ **(e)** __(iii)__ **(f)** __(iv)__

Answer: Graph **(a)** is a parabola translated along the y-axis. Consequently, the equation of the corresponding function should have no y present. Graph **(b)** is a saddle. The cross sections for $x = k$ are parabolas opening up while for $y = k$, they are parabolas opening down. This agrees with the function $f(x,y) = y^2 - x^2$. Graph **(c)** is also a saddle, but now the parabolas open up for the traces with $y = k$ and open down when $x = k$. This agrees with the function $f(x,y) = x^2 - y^2$. Graph **(d)** is the lower half of the cone $z^2 = x^2 + y^2$, thus $z = -\sqrt{x^2 + y^2}$. Graph **(e)** is an upside down bowl, which corresponds to $f(x,y) = -(x^2 + y^2)$, while graph **(f)** is a plane described by the only linear function.

2. Multiple Choice: Choose the best answer available (no work required). (5 pts. each)

(a) Which equation represents a sphere with center $(-1, 0, 2)$ passing through the point $(3, 1, -1)$?

(i) $x^2 + 2x + y^2 + z^2 + 4z = 13$,
(ii) $x^2 + 2x + y^2 + z^2 - 4z = 21$,
(iii) $x^2 + 2x + y^2 + z^2 - 4z = 26$,
(iv) $x^2 - 2x + y^2 + z^2 + 4z = 21$,
(v) $x^2 + y^2 + z^2 = 26$.

Answer: (ii). The radius of the sphere is the distance between $(-1, 0, 2)$ and $(3, 1, -1)$, which is $\sqrt{(3 - -1)^2 + (1 - 0)^2 + (-1 - 2)^2} = \sqrt{26}$. Therefore, the equation of the sphere is $(x + 1)^2 + y^2 + (z - 2)^2 = 26$, which simplifies to $x^2 + 2x + y^2 + z^2 - 4z = 21$.

(b) Suppose that $\mathbf{v}$ and $\mathbf{w}$ are two non-parallel vectors in $\mathbb{R}^3$. Which of the following statements is **NOT** true?

(i) $\mathbf{v} \times \mathbf{w}$ is a vector.
(ii) $\mathbf{v} \times \mathbf{v} = 0$.
(iii) $\mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{v}$.
(iv) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.
(v) $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

Answer: (iii). The cross product is anti-commutative because of the right-hand rule, so $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$. The equation $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ simply says that $\mathbf{w}$ is orthogonal to the cross product of $\mathbf{v}$ and $\mathbf{w}$, which is true by definition of the cross product.

(c) The graph of the quadric surface $4x^2 - y^2 - z^2 = 1$ is

(i) an ellipsoid,
(ii) an elliptic paraboloid,
(iii) a cone,
(iv) a hyperboloid of one sheet,
(v) a hyperboloid of two sheets.

Answer: (v). Taking cross sections $x = k$ yields $y^2 + z^2 = 4k^2 - 1$, which means the traces are circles, but only when $4k^2 - 1 > 0$. This last inequality is satisfied for $k > 1/2$ or $k < -1/2$. Consequently, there are two pieces of the surface, with a gap for $-1/2 < x < 1/2$. A hyperboloid of two sheets is the only choice with this property.

(d) The path of $\mathbf{r}(t) = \cos 3t\,\mathbf{i} + 4\,\mathbf{j} + \sin 3t\,\mathbf{k}$ traces out a

(i) helix (slinky) in xyz-space,
(ii) line in xyz-space,
(iii) circle of radius 3 in the plane $x + 4y + z = 0$,
(iv) circle of radius 3 in the plane $y = 4$,
(v) circle of radius 1 in the plane $y = 4$.

Answer: (v). This one was a little tricky. The y-component of the curve is just 4 (not $4t$). Consequently, the curve lies in the plane $y = 4$. Since $(x(t))^2 + (z(t))^2 = 1$ for all t, the curve is a circle of radius 1.

3. Given the vectors $\mathbf{a} = \mathbf{i} + 7\,\mathbf{k}$ and $\mathbf{b} = -2\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$, compute each of the following quantities: (15 pts.)

(a) $3\mathbf{a} - 2\mathbf{b}$
Answer: $3\mathbf{a} - 2\mathbf{b} = 3\,\mathbf{i} + 21\,\mathbf{k} - 2(-2\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}) = 3\,\mathbf{i} + 21\,\mathbf{k} + 4\,\mathbf{i} - 6\,\mathbf{j} - 4\,\mathbf{k} = 7\,\mathbf{i} - 6\,\mathbf{j} + 17\,\mathbf{k}$

(b) $|\mathbf{a} - \mathbf{b}|$
Answer: $|\mathbf{a} - \mathbf{b}| = |3\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}| = \sqrt{9 + 9 + 25} = \sqrt{43}$

(c) $\text{proj}_{\mathbf{a}}\mathbf{b}$ (the vector projection of $\mathbf{b}$ onto $\mathbf{a}$)
Answer: Using $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$, we have

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{-2 + 14}{1 + 49}\right)(\mathbf{i} + 7\,\mathbf{k}) = \frac{12}{50}\,\mathbf{i} + \frac{84}{50}\,\mathbf{k} = \frac{6}{25}\,\mathbf{i} + \frac{42}{25}\,\mathbf{k}$$

4. Below are equations for two different pairs of planes. For one pair of equations, the planes are parallel; for the other pair of equations, the planes intersect. Identify which pair is which, and find the acute angle (to the nearest degree) between the two planes that intersect. (10 pts.)

$$\textbf{(i)}\quad \begin{aligned} x - y + 2z &= 10 \\ 3x + y - 2z &= 15 \end{aligned} \qquad\qquad \textbf{(ii)}\quad \begin{aligned} x - y + 2z &= 10 \\ -4x + 4y - 8z &= 15 \end{aligned}$$

Answer: The key is to consider the normal vectors for each plane. For the first pair of planes, we have $\mathbf{n_1} =< 1, -1, 2 >$ and $\mathbf{n_2} =< 3, 1, -2 >$. Since these two vectors are not scalar multiples of each other, they are not parallel, and thus the two planes are not parallel (and must intersect in a line.) On the other hand, the second pair of planes has $\mathbf{n_1} =< 1, -1, 2 >$ and $\mathbf{n_2} =< -4, 4, -8 >$. Since $\mathbf{n_2} = -4\mathbf{n_1}$, the normal vectors are parallel and thus the two planes are parallel (they are not the same plane because $15 \neq -4 \cdot 10$.)

To find the acute angle of intersection between the first pair of planes, we find the angle of intersection of the normals (by definition). Using the dot product formula $\mathbf{n_1} \cdot \mathbf{n_2} = |\mathbf{n_1}||\mathbf{n_2}| \cos\theta$, we find $-2 = \sqrt{6} \cdot \sqrt{14}\, \cos\theta$, which implies

$$\cos\theta = \frac{-2}{\sqrt{6} \cdot \sqrt{14}} = \frac{-1}{\sqrt{21}}.$$

Thus, $\theta = \cos^{-1}(-1/\sqrt{21}) \approx 103°$. To find the acute angle we subtract this from 180°, which yields an angle of 77°.

5. Consider the three points $P = (-1, 2, 2), Q = (5, -1, 2)$ and $R = (1, 0, 3)$. (20 pts.)

(a) Find the equation of the plane containing all three points.

Answer: To find the equation of the plane, we first need a normal vector to the plane. This can be obtained by taking the cross product of any two vectors in the plane. The vectors $\overrightarrow{PQ} =< 6, -3, 0 >$ and $\overrightarrow{PR} =< 2, -2, 1 >$ are each in the plane. The vector $\overrightarrow{PQ}$ is found by subtracting the coordinates of P from the coordinates of Q (i.e., $\overrightarrow{PQ} = Q - P$). The cross product is then computed as

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left| \begin{array}{ccc:cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 6 & -3 & 0 & 6 & -3 \\ 2 & -2 & 1 & 2 & -2 \end{array} \right| = -3\,\mathbf{i} - 12\,\mathbf{k} + 6\,\mathbf{k} - 6\,\mathbf{j} = -3\,\mathbf{i} - 6\,\mathbf{j} - 6\,\mathbf{k}.$$

Thus, a normal vector to the plane is $\mathbf{n} =< -3, -6, -6 >$. This means the plane has the equation $-3x - 6y - 6z = d$, where d is some constant. To find d we can plug in any of the three given points. Using point R (the easiest choice), we find that $-3(1) - 6(3) = d$, so $d = -21$. Dividing both sides by -3, the equation of the plane is $x + 2y + 2z = 7$. Note that all three points satisfy the equation.

(b) Find the area of the triangle PQR.

Answer: The area of triangle PQR is

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}| < -3, -6, -6 > | = \frac{1}{2}\sqrt{9 + 36 + 36} = \frac{9}{2}.$$

6. Consider the vector function $\mathbf{r}(t) =< e^{2t}, \sqrt{3 - 2t}\,, t\sin(\pi t) >$. (17 pts.)

(a) Find the domain of the vector function.

Answer: The domain of the vector function is the intersection of the domains of each component. The first and third component are defined for all real numbers, consequently, the domain of the function is simply the domain of the second component. In order for the square root to be defined, we must have $3 - 2t \geq 0$, which simplifies to $2t \leq 3$ or $t \leq 3/2$. Thus the domain of the function is $(-\infty, 3/2]$ or $t \leq 3/2$.

(b) Find the unit tangent vector $\mathbf{T}(t)$ at the point where $t = 0$.

Answer: To find the unit tangent vector we first compute $\mathbf{r}'(t)$. Using the chain rule and the product rule, we have

$$\mathbf{r}'(t) =< 2e^{2t}, \frac{1}{2}(3-2t)^{-1/2}\,(-2), \sin(\pi t) + t\cos(\pi t)\cdot\pi >=< 2e^{2t}, \frac{-1}{\sqrt{3-2t}}, \sin(\pi t) + \pi t\cos(\pi t) > .$$

To find the tangent vector at the point where $t = 0$, we plug in $t = 0$ to obtain $\mathbf{r}'(0) =< 2, -1/\sqrt{3}, 0 >$. To make this vector a unit vector, we divide by its length, $|\mathbf{r}'(0)| = \sqrt{4 + 1/3} = \sqrt{13/3}$. Hence,

$$\mathbf{T}(0) = \frac{1}{\sqrt{13/3}} < 2, \frac{-1}{\sqrt{3}}, 0 > = \frac{\sqrt{3}}{\sqrt{13}} < 2, \frac{-1}{\sqrt{3}}, 0 > = < \frac{2\sqrt{3}}{\sqrt{13}}, \frac{-1}{\sqrt{13}}, 0 > .$$

(c) Find parametric equations for the tangent line to the parametrized curve at the point $(e^2, 1, 0)$.

Answer: To find parametric equations of a line we need a point and a direction. The point is given. To find the direction we can use the derivative $\mathbf{r}'(t)$. However, we need to know what *time* the curve passes through the given point. The first component, e^{2t}, equals e^2 when $t = 1$. Checking the other components, we see that $\mathbf{r}(1) = (e^2, 1, 0)$, so that $t = 1$ is the time when the curve reaches the given point. The direction of the tangent line is then given by $\mathbf{r}'(1) =< 2e^2, -1, -\pi >$. Thus, parametric equations describing the tangent line are

$$\begin{aligned} x &= e^2 + 2e^2 t \\ y &= 1 - t \\ z &= -\pi t. \end{aligned}$$