Multivariable Calculus, Spring 2005

Computer Project #3

Optimization: Celestial Mechanics and Least Squares

DUE DATE: Friday, April 8th, in class.

1 Introduction

This project focuses on two particular applications of multivariable calculus using optimization. There are plenty of problems where extrema of functions are required in order to find an optimal answer to a physical question. Here you will re-discover one of the first solutions to the three-body problem in Celestial Mechanics and help a field biologist estimate the population of rare toads. The mathematics required is finding and classifying critical points of functions of two variables. We will make use of the computer software MAPLE to visualize functions, draw contour diagrams, locate extrema and simplify our computations.

It is **required** that you work in a group of two or three people, your original lab team. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced. Please turn in one report per group, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully and neatly, **writing in complete sentences.**

The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. It is also possible to write your answers within your MAPLE worksheet using the text option. Your presentation is important and I should be able to clearly read and understand what you are saying. Your report should provide answers to each of the questions below.

2 Celestial Mechanics: The planar circular restricted three-body problem

Concerning the solution to the three-body problem (studying for example the motion of the Earth, Sun and Moon), Isaac Newton once frustratingly remarked to the astronomer John Machin that "... his head never ached but with his studies on the moon."[1] Although Newton successfully used his theory of Calculus to prove that Kepler's empirical laws were in fact accurate (eg. the Earth travels on an elliptical orbit with the Sun at one of the focal points), he was unable to find a solution to the motion of three bodies in space, the so-called three-body problem. Although it was posed centuries ago, we have made little progress in understanding this question: Given three masses in space with their initial positions and initial velocities, what is the motion of the bodies over time if the only force acting upon them is their mutual gravitation? To this day, the three-body problem remains an open question and an active area of research in the fields of mathematics, physics, astronomy and space travel. Note that this April is Mathematics Awareness Month, with the theme *Mathematics and the Cosmos*, and the poster for the event describes an interplanetary superhighway made up of special solutions to the three-body problem.

One way of approaching the complicated three-body problem is to assume we know the motion of two massive bodies (say the Earth and the Sun) and then try to ascertain the motion of a third, infinitesimal mass (say the moon or a satellite). Let us assume that the larger bodies, called **primaries**, are on circular orbits about their center of mass and that their motion lies in a plane. The third infinitesimal mass, being so small, does not effect the motion of the larger bodies, whose orbits remain circular. Investigating the motion of the third mass subject to the gravitational attraction from the other two is called the **planar circular restricted three-body problem**. In this exercise, you will find some of the simplest (and earliest) solutions to this problem.

Using Newton's law of gravitation, the force between two bodies is proportional to the product of their masses and inversely proportional to the square of the distance between them. Using $F = ma$, we can set up a system of differential equations (a is a second derivative of position) that governs the motion of the infinitesimal particle. We can choose a rotating coordinate frame so that in the new coordinates, the position of the first and second mass appear fixed.

Let μ be a positive real number (a parameter) with $\mu < 1$. Denote $m_1 = \mu$ and $m_2 = 1 - \mu$ as the masses (normalized so that $m_1 + m_2 = 1$) of the two large bodies, and fix their positions to be in the plane at the points $\mathbf{q}_1 = (1 - \mu, 0)$ and $\mathbf{q}_2 = (-\mu, 0)$. This is done so that their center of mass $(m_1 q_1 + m_2 q_2)/(m_1 + m_2)$ is at the origin (check it). Let the position of the third infinitesimal mass be given by (x, y), where each variable is really a function of time. The motion of the third infinitesimal mass involves what is called the **effective potential**, given by

$$V(x,y) = \frac{1}{2}(x^2 + y^2) + \frac{\mu}{\sqrt{(x-1+\mu)^2 + y^2}} + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2}} \,. \tag{1}$$

Note that the denominators of the two fractions in equation (1) give the distances between the large bodies and the small infinitesimal mass. This comes from Newton's inverse square law.

It turns out that the critical points (extrema) of the effective potential $V(x, y)$ are **equilibrium solutions** of the differential equations governing the motion of the third body. This is a special solution to the system of differential equations in which the third body is at rest with respect to the other two. Of course the whole system is going around in a circle about the origin so the third body would be moving in a circular orbit just like the others. (Think of three points on a spinning record player: Seen from the record, the points are always the same distance apart, while seen from above they are moving in circles about the center of the record.) These solutions are called **libration points** in classical celestial mechanics literature. They were discovered by the great mathematicians Lagrange and Euler in the mid 18th century and were some of the first solutions found in the three-body problem.

1. Begin by setting $\mu = 1/2$, so $m_1 = m_2 = 1/2$, that is, the primaries have equal mass. One might expect to find some nice symmetry in this case.

   (a) Setting $\mu = 1/2$, what is the domain of $V(x, y)$? Where is $V(x, y)$ undefined and what happens to the effective potential as you approach these points? Note: You could use MAPLE and the `plot3d` command to view a graph of the function. The `view=-3..3` option is helpful here — you pick an appropriate range.

   (b) Compute the partial derivatives V_x and V_y BY HAND. Show that $V_x = 0$ at any point on the y-axis and $V_y = 0$ at any point on the x-axis in the domain.

   (c) Find all of the critical points of $V(x, y)$ as accurately as possible and classify them as maxima, minima or saddles.

Hint: Use the result from the previous question. The command `fsolve` can be used to solve a system of equations in two variables. For example, the command

```
fsolve({x^2 + y^2 = 1, 2*x - y = 0}, {x,y},x=0..1,y=0.3..1);
```

simultaneously solves the system $x^2+y^2=1, y=2x$ in the region $0 \le x \le 1, 0.3 \le y \le 1$. It is very important to specify a small region which contains a solution so that the computer can accurately find an answer. Since there are several extrema, you will need to specify this region carefully for each one. Use MAPLE first to find the approximate locations of each extrema.

(d) Draw a graph in the xy-plane containing all the critical points as well as the two positions of the large primaries with masses $m_1 = m_2 = 1/2$. What can you say about the location of each critical point with respect to the two large primaries, ie. what shape is formed by the three bodies? Describe any symmetries you notice in your figure.

2. Now assume that $\mu = 1/10$, so $m_1 = 1/10$ and $m_2 = 9/10$, that is, the second primary is nine times larger than the first.

(a) Setting $\mu = 1/10$, what is the domain of $V(x,y)$? Where is $V(x,y)$ undefined and what happens to the effective potential as you approach these points?

(b) Compute the partial derivatives V_x and V_y BY HAND. Is it still true that $V_x = 0$ at any point on the y-axis and $V_y = 0$ at any point on the x-axis in the domain? Explain.

(c) Find all of the critical points of $V(x,y)$ as accurately as possible and classify them as maxima, minima or saddles.

Hint: Use the result from part **(b)** as well as the techniques described in Question #1.

(d) Draw a graph in the xy-plane containing all the critical points as well as the two positions of the large primaries. What can you say about the location of each critical point with respect to the two large primaries, ie. what shape is formed by the three bodies? Compare your answers here to those in the previous question, part **(d)**.

Extra Credit: Prove that for any value of $\mu \in (0,1)$, there are always five libration points. Where are they?

3 Least Squares: How to fit a line to a set of data

Another important optimization problem involves fitting a set of data to a prescribed curve. Quite often the data we record in the real world is not perfectly linear or quadratic or cubic or exponential, etc. However, it may be well-approximated by a particular line or curve. This exercise gives one way in which to find the "best" such curve, called **the method of least squares** (see pp. 713-714 of the course text [2].)

Suppose that a biologist working in the field for several months records the following measurements for the number of rare toads, where t is measured in months and n represents the number of toads observed in the hundreds. The data is plotted in Figure 1. The value $n = 20$ at $t = 0$ is an assumption on the part of the biologist based on a previous expedition.

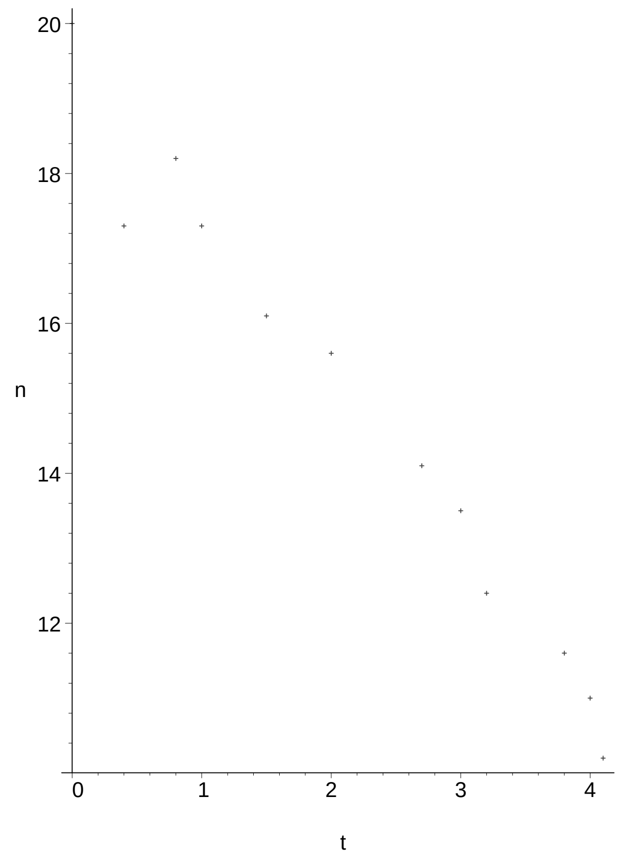

Figure 1: The data collected by a biologist on the number of rare toads n in hundreds versus time t in months.

t in months	n in hundreds
0	20
0.4	17.3
0.8	18.2
1	17.3
1.5	16.1
2	15.6
2.7	14.1
3	13.5
3.2	12.4
3.8	11.6
4	11
4.1	10.2

Let (t_i, n_i) represent the i-th data point from the table. (So i ranges between 1 and 12.) We seek to find the best line such that the sum of the squares of the vertical distance between each point and the line is as small as possible. In other words, suppose that $n = mt + b$ is the equation of the best fitting line. Then the point on the line directly above or below (t_i, n_i) has coordinates $(t_i, mt_i + b)$. We want the distance squared between all such pairs of points to be small, that is, we seek the unknown constants m and b so that

$$F(m,b) \;=\; \sum_{i=1}^{12} (n_i - (mt_i + b))^2$$

is as small as possible.

1. Plug in the twelve data points above to form the function $F(m,b)$. You can use MAPLE to do the algebra for you, the hard part is typing in the data correctly. Go slowly! The command `expand( )` will expand and simplify any expression inside the parentheses. For example, `expand( (x-y)^2);` gives the polynomial $x^2 - 2xy + y^2$. You should obtain an expression with only m and b in it, a function of two variables! Give your final result $F(m,b)$.

2. Compute the first partial derivatives of $F(m,b)$ with respect to m and b and find the critical point. This gives the precise value of the slope and n-intercept of the best least squares line. Plot this line on the graph with the data and turn it in. Give the equation of the best least squares line.

3. Compute the second-order partials and show that the critical point you found in part 2. is indeed a minimum.

4. Based on your linear approximation to the biologists' data, when should we expect the toad species to go extinct?

5. Explain why we did a least "squares" fit. What happens if you used the vertical distance between the point and the line without squaring? Where does the mathematical argument break down?

Extra Credit: Do Project 2 on pp. 733 - 734 of the text [2] in full detail to derive the general formula for fitting a line to a set of data using least squares.

References

[1] Barrow-Green, J. Poincaré and the Three Body Problem, *History of Mathematics*, vol. 11, American Mathematical Society, pp. 15., 1997.

[2] McCallum, W., Hughes-Hallett, D., et al. Multivariable Calculus, 3rd ed., John Wiley and Sons, Inc., New York, 2002.

[3] Meyer, K. and Hall, G. Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, Springer-Verlag, New York, 1992.