

# Multivariable Calculus, Spring 2005

## Computer Project #2

### Tangent Planes and the Dot Product

**DUE DATE:** Friday, March 4th, in class.

## 1 Introduction

This project focuses on plotting the tangent plane to a function of two variables  $f(x, y)$  and using the dot product to help a dating service make the right match. The discussion on tangent planes began with the **MAPLE Worksheet on Tangent Planes** we started on Friday, Feb. 18. The tangent plane to the graph of a function  $f$  at the point  $(a, b)$  is the graph of the best linear approximation to  $f$  at  $(a, b)$ . This is a generalization of the tangent line from Calculus 1, where the tangent line gave the best linear approximation to a function of one variable. We will make use of the computer software MAPLE to visualize tangent planes to graphs, in particular focusing on tangent planes to points which are extrema (maxs, mins or saddle points).

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced. Please turn in one report per group, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully and neatly, **writing in complete sentences**.

The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Your report should provide answers to each of the questions below.

## 2 Tangent Planes

Recall that the equation of the tangent plane for  $f(x, y)$  at the point  $(a, b)$  is given by

$$z = f_x(a, b)x + f_y(a, b)y + c \quad (1)$$

where  $c$  is determined by making sure the plane passes through the point  $(a, b, f(a, b))$ . Another formula often used is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (2)$$

giving the specific value of  $c$ . This formula is reminiscent of the start of a Taylor series! *Note:* Details, explanations and examples of useful commands regarding tangent planes can be found in the **MAPLE Worksheet on Tangent Planes** we started on Friday, Feb. 18. (This worksheet is available on the course webpage as well.)

1. For the function  $f(x, y) = 1 - x^2 - y^2$ , find the equation of the tangent plane at the point  $(3, -2)$ . Plot the tangent plane and the function on the same graph (choose a good domain to plot over) and turn in this plot. Find the equation of the tangent plane at the point  $(0, 0)$ .

What is significant about this point? Plot the tangent plane and the function on the same graph (choose a good domain to plot over) and turn in this plot.

- For the function  $g(x, y) = y^3 - 12y + 2x^2 + 4x + 4$ , find the equation of the tangent plane at the point  $(-1, 2)$ . What is significant about this point? Plot the tangent plane and the function on the same graph and turn in this plot. Compare with the previous problem.
- For the function  $h(x, y) = \sin(xy)$ , find the equation of the tangent plane at the point  $(0, 0)$ . What kind of point is  $(0, 0)$ ? Plot the tangent plane and the function on the same graph (choose a good domain to plot over) and turn in this plot.
- Based on Questions 1–3, what can you conclude about the tangent plane to a maximum, minimum or saddle? How is this similar to what we know from Calc 1 when considering the tangent line at an extremum? In general, given an arbitrary function  $f(x, y)$ , how do we locate the extrema? Compare this with how we find extrema for functions of one variable.
- Define a MAPLE function called `tanplane(a,b,h)` which plots both the function  $f(x, y) = 1 - x^2 - y^2$  and its tangent plane at the point  $(a, b)$  over the plot range  $a - h \leq x \leq a + h, b - h \leq y \leq b + h$ . For example, when you type `tanplane(-1,0,2)`, you should get a 3d plot of  $f(x, y)$  and its tangent plane at the point  $(-1, 0)$  drawn over the plot range of  $-3 \leq x \leq 1, -2 \leq y \leq 2$ . The first part of your command should look like

```
tanplane := (a,b,h) -> plot3d({...}, ... );
```

You must fill in the important information inside the `plot3d` command (this will contain the input variables `a`, `b` and `h`.) Once you get your command working, use it to view various tangent planes on the graph of  $f(x, y)$  to get a better understanding of the function. The only item that needs to be turned in for this question is the definition of your function `tanplane(a,b,h)`. No graphs should be turned in for this question.

- Repeat Problem 5 for the function  $h(x, y) = \sin(xy)$ . Again, the only item that needs to be turned in for this question is the definition of your function `tanplane(a,b,h)`. No graphs should be turned in for this question.

### 3 Applications of the Dot Product: A Dating Service

The dot product between two vectors  $\vec{v} \cdot \vec{w}$  can be defined regardless of the dimension of the space the vectors live in. This is not true for the cross product. If  $\vec{v}$  has components  $(v_1, v_2, \dots, v_n)$  and  $\vec{w}$  has components  $\vec{w} = (w_1, w_2, \dots, w_n)$  (so  $\vec{v}, \vec{w} \in \mathbb{R}^n$ ), then we define the dot product as

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n.$$

This is just a generalization of the formula for vectors with 3 components. The same algebraic properties that held for the standard dot product in 3-d hold here as well (commutative, distributive, etc.). As before, the length of a vector can be written in terms of the dot product:

$$||\vec{v}||^2 = \vec{v} \cdot \vec{v}$$

But what about the geometric properties? If  $\vec{v}, \vec{w} \in \mathbb{R}^{20}$ , it's hard to imagine the cosine of the angle between two vectors in 20 dimensional space. However, we can use the equation

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos \theta$$

to **define** the angle  $\theta$  between two vectors in  $\mathbb{R}^n$ . In other words,  $\theta$  can be found by solving the equation

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| ||\vec{w}||} \quad (3)$$

for  $\theta$ . The only problem with this definition is that we do not **a priori** know that the right-hand side of equation (3) is actually between  $-1$  and  $1$ . This must be true, otherwise the equation may be undefined. We will prove this with a few exercises.

Suppose that  $\vec{v}$  and  $\vec{w}$  are any two vectors in  $\mathbb{R}^n$  and define a function  $f(t)$  of a single variable  $t$  by

$$f(t) = (\vec{v} + t\vec{w}) \cdot (\vec{v} + t\vec{w})$$

1. Notice that  $f(t)$  outputs real numbers. Show that  $f(t) \geq 0$  for all  $t$ .
2. Using the standard algebraic properties of the dot product, expand  $f(t)$  into a quadratic polynomial in  $t$ . (These properties are on page 622 of the text by McCallum, et. al.)
3. Knowing that  $f(t) \geq 0$  for all  $t$ , compute the discriminant of the quadratic polynomial to show that

$$|\vec{v} \cdot \vec{w}| \leq ||\vec{v}|| ||\vec{w}|| \quad (4)$$

Recall that the discriminant of  $ax^2 + bx + c$  is  $b^2 - 4ac$ . The inequality (4) is called the **Cauchy-Schwartz** inequality.

This clever proof guarantees that the right-hand side of equation (3) is actually between  $-1$  and  $1$ , and so the definition of  $\cos \theta$  makes sense. One of the useful applications of this fact is that it provides us with a measure of closeness for two vectors in  $\mathbb{R}^n$ . If  $\theta$  is small, then  $\vec{v}$  and  $\vec{w}$  are close together. If  $\theta$  is large, then  $\vec{v}$  and  $\vec{w}$  are far apart.

### Computer Dating Service

Suppose that Eliza has decided to let a computer dating service match her up with one of four eligible bachelors, Aaron, Bart, Carl and Doug. The method which the dating service uses is to have participants rate how much they like 10 different ice-cream flavors on a scale of 1 to 10 (1 for flavors you detest, 10 for flavors you love). By pairing up people with similar ice-cream preferences, the dating service figures that this is as good a criterion as any for matching up future mates.

The table below shows how each contestant ranked the 10 flavors. Each column of the table can be thought of as a vector in  $\mathbb{R}^{10}$  representing the preferences of each contestant.

Flavors	Eliza	Aaron	Bart	Carl	Doug
vanilla	5	9	2	5	10
chocolate	7	4	8	6	8
strawberry	4	2	3	6	7
mocha almond	9	9	8	7	5
peppermint	8	5	3	5	6
butter pecan	7	10	8	8	7
oreo cookie	6	3	8	7	2
coffee	2	4	5	7	5
peaches 'n cream	4	5	3	6	8
cinnamon	8	8	4	8	7

4. Note that the length of any column vector is a measurement of how much that particular individual likes ice-cream in general, as the higher numbers indicate they prefer many flavors, not just one or two. By computing the lengths of the vectors for each contestant, place them in order based on how much ice-cream they like.

*Hint:* MAPLE has a command called `dotprod(v,w)` which computes the dot product of the vectors  $\vec{v}$  and  $\vec{w}$ , as long as both have the same number of components (otherwise the dot product isn't defined). To use this command you need to load in a linear algebra package by typing `with(linalg):`. A vector can be created in MAPLE as a list, for example, `v := [1,2,3,4,5];`, which assigns the vector  $(1, 2, 3, 4, 5)$  to  $\vec{v}$ .

5. Compute the cosine of the angle between the vectors for each of the male contestants and Eliza to determine who the dating service matches up with Eliza. Who finishes second, third and fourth?