Multivariable Calculus, Feb. 18, 2005

MAPLE Worksheet on Tangent Planes

The focus of this worksheet is to use MAPLE to see how the tangent plane to a differentiable function is the best linear approximation to the function. Recall from Calc 1 that the tangent line to a function of one variable gives the best linear approximation to the graph. In fact, as you zoom in on the curve at a particular point P, the curve and tangent line become indistinguishable. This is easy to demonstrate using MAPLE.

1 A Calc 1 Example

Suppose that $f(x) = \sin x$. Calculate the equation of the tangent line to the graph when $x = \pi/3$. Do this without the computer. To plot this line $mx + b$ (you have found m and b) along with the graph of $\sin x$, you type:

```
plot([m*x + b,sin(x)],x=0..Pi, color=[red,blue]);
```

This should draw the curve in blue and the tangent line in red. Make sure your tangent line is actually tangent to the graph at $x = \pi/3$. Recall that `Pi` is how you type in π and `sqrt(x)` is how you get $\sqrt{x}$.

Next, we want to zoom in on the graph near $x = \pi/3$. To do this without retyping the `plot` command over and over again, we define a zooming command by typing the following:

```
zoomplot:= h -> plot([m*x + b,sin(x)], x=Pi/3-h..Pi/3+h, color=[red,blue]):
```

This creates the command `zoomplot(h)` which you can now use to easily zoom in on the graph near $x = \pi/3$. For example, if you want the plot range to be centered about $\pi/3$ with a distance of 0.3 to either side, you type `zoomplot(0.3)`. This should give you the graph and the tangent line drawn on the x range of $[\pi/3 - 0.3, \pi/3 + 0.3]$. Use the `zoomplot` command to zoom in on your graph until you see that the tangent line and curve become nearly identical.

Two important notes: First, you can define a constant to a letter by typing:

```
b := Pi/10 - 3.6739;
```

for example. This way you can refer to the constant as b each time rather than having to repeatedly retype it. Second, your zoomplot command only works for the example given, $\sin x$. To do a different function, tangent line and point $x = a$, you must return to the definition of `zoomplot` and type in the new functions you want.

2 Tangent Planes

Now we will create the equivalent approximation but to a function of two variables. Consider the function $f(x, y) = 1 - x^2 - y^2$. Plot the function over the range $-4 \le x \le 4, -4 \le y \le 4$. We would like to zoom in on the function around the point $(-2, 1)$. As before, we can define a zooming function by typing:

```
zoomplot3d:= h -> plot3d(1-x^2-y^2, x=-2-h..-2+h,y=1-h..1+h,axes=boxed):
```

To draw the graph over the domain $-3 \leq x \leq -1, 0 \leq y \leq 2$, for example, type `zoomplot3d(1)` giving one units length to either side of $x = -2$ and $y = 1$. Zoom in on the graph near $(-2, 1)$ using the `zoomplot3d(h)` command. What do you notice?

Now plot the linear function $L(x, y) = 4x - 2y + 6$ over the range $-3 \leq x \leq -1, 0 \leq y \leq 2$. Compare with your zooms of the function $f(x, y)$. What do you notice? This plane is called the **tangent plane** to the function $f(x, y)$ at the point $(-2, 1)$ and is the analog of the tangent line to a one-variable function at a point $x = a$. **STOP:** Before reading further, try and derive the equation of this plane using partial derivatives and the equation for a linear function discussed in class.

Just as for functions of one variable, we use partial derivatives to find the particular slopes of the tangent plane. Recall that $L(x, y) = mx + ny + c$ gives the equation of a plane, where m is the slope in the x-direction, n gives the slope in the y-direction and c is the z-intercept. When finding the equation of a tangent plane to a function $f(x, y)$, m is obtained by computing the partial derivative of f with respect to x and evaluating at the point in question. After all, this tells us the slope of the graph in the x-direction for fixed y. Likewise, n is obtained by computing the partial derivative of f with respect to y. This tells us the slope of the graph in the y-direction for fixed x.

To obtain m in our example, compute $f_x(-2, 1)$, the partial derivative of f with respect to x evaluated at the point $(-2, 1)$. This gives $m = f_x(-2, 1) = 4$. Similarly, to obtain n for our example, compute $f_y(-2, 1)$, the partial derivative of f with respect to y evaluated at the point $(-2, 1)$. This gives $n = f_y(-2, 1) = -2$. Finally, the tangent plane should pass through the point on the graph $(-2, 1, f(-2, 1)) = (-2, 1, -4)$. Use this point to find the z-intercept by plugging it into $z = 4x - 2y + c$ and solving for c. This gives $c = 6$. Thus, the equation of the tangent plane to $f(x, y) = 1 - x^2 - y^2$ at the point $(-2, 1)$ is given by

$$z = 4x - 2y + 6.$$

Of course, you probably can't wait to see the graph and the tangent plane together! You can achieve this by typing:

```
plot3d({4*x-2*y+6,1-x^2-y^2},x=-4..4,y=-4..4,axes=framed);
```

Click on your graph and rotate around to see where the two surfaces touch. Zoom in on the point $(-2, 1)$ to see the surfaces becoming more and more the same.

In sum, the equation of the tangent plane for $f(x, y)$ at the point (a, b) is given by

$$z = f_x(a, b)\, x + f_y(a, b)\, y + c \tag{1}$$

where c is determined by making sure the plane passes through the point $(a, b, f(a, b))$. Another formula often used is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{2}$$

giving the specific value of c. Note that this formula is reminiscent of the start of a Taylor series!

3 Exercises:

1. For the function $f(x, y) = 1 - x^2 - y^2$, compute the tangent plane at the point $(0, 0)$. What is significant about this point? Plot the tangent plane and the function on the same graph.

2. For the function $g(x, y) = y^3 - 12y + 2x^2 + 4x + 4$, compute the tangent plane at the point $(-1, 2)$. What is significant about this point? Plot the tangent plane and the function on the same graph. Compare with the previous problem.

3. For the function $h(x, y) = \sin(xy)$, compute the tangent plane at the point $(0, 0)$. What kind of point is $(0, 0)$? Plot the tangent plane and the function on the same graph, in an appropriate plot range.

4. Based on Questions 1–3, what can you conclude about the tangent plane to a maximum, minimum or saddle? How is this similar to what we know from Calc 1 when considering the tangent line at an extremum? Given a function $f(x, y)$ in general, how do we locate the extrema? Compare this with how we find extrema for functions of one variable.

5. Define a MAPLE function called `tanplane(a,b,h)` which plots both the function $f(x, y) = 1 - x^2 - y^2$ and its tangent plane at the point (a, b) over the plot range $a - h \leq x \leq a + h, b - h \leq y \leq b + h$. For example, when you type `tanplane(-1,0,2)`, you should get a 3d plot of $f(x, y)$ and its tangent plane at the point $(-1, 0)$ drawn over the plot range of $-3 \leq x \leq 1, -2 \leq y \leq 2$. The first part of your command should look like

```
tanplane := (a,b,h) -> plot3d({...}, ...  );
```

   You must fill in the important information inside the `plot3d` command. Once you get your command working, use it to view various tangent planes on the graph of $f(x, y)$ to get a better understanding of the function.

6. Repeat Problem 5 for the function $h(x, y) = \sin(xy)$.