Multivariable Calculus, Spring 2001

Computer Project #1

Graphing functions of several variables

DUE DATE: Friday, Sept. 28th, in class.

This project explores ways of visualizing functions of two variables, $f(x, y)$. The two main graphical methods of representing such a function are, in 3-d, by plotting the set of points $(x, y, f(x, y))$ in xyz-space, or in 2-d, by drawing a contour plot of level curves in the xy-plane. We have discussed both of these ideas extensively in class. In this project you will search for a secret message, find the maximum and minimum heights of various functions, analyze contour plots in detail, become an ant traveling across a hot plate and discover the limitations of computers. Using the software package Maple (remember to type `xmaple`), you will make use of commands such as `plot3d, contourplot, contourplot3d` and `densityplot` to obtain graphs for different functions. Recall that to load the commands for contour plots you must type `with(plots):`

The goal of the project is for you to explore and learn about this aspect of multivariable calculus and to communicate your understanding and discoveries in a well-written and coherent report. Some of the problems are difficult and are designed to challenge you, perhaps in motivation for future research. It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced.

The project should be typed although you do not have to typeset your mathematical notation. For example, you can leave space for a graph, computations, tables, etc. and then write it in by hand later. You can also include graphs or computations in an appendix at the end of your report. Your presentation is important and I should be able to clearly read and understand what you are saying. Spelling mistakes and sentence fragments, for example, should not occur. Only one project per group need be submitted.

Your report should provide answers to each of the following questions. Be sure to answer all of the questions asked. Read carefully. You do not have to include an introduction in your report, although I would like a **conclusion** which states what you learned from the project and further questions you might like to investigate. The bulk of your report should consist of coherent answers to the questions below.

Acknowledgments: Ideas and equations for this computer project were borrowed from the lab project *Surfaces, contours, gradients and applications* by Kristian Sandberg, Dept. of Applied Math., University of Colorado, Boulder, September, 1999 and from *Multivariable Calculus: Collaborative Learning Workbook* by David B. Damiano and Margaret N. Freije, Dept. of Math. and Comp. Sci., College of the Holy Cross, June, 2001.

1. Using `plot3d`, draw a 3-d plot of the function $f(x, y) = \cos(4x)\, e^{-(x^2+y^2/2)} + e^{-3((-x+0.5)^2+y^2/2)}$ over the region $-3 < x < 3$, $-5 < y < 5$. This function can be assigned to f by typing

```
f := (x,y) -> cos(4*x)*exp(-(x^2+y^2/2)) + exp(-3*((-x+0.5)^2 + y^2/2);
```

Anytime you want to use this function in a command, you just type `f(x,y)` in place of where the expression would normally go. There is a secret message in the graph which can only be found if you restrict the output values (z-values) to be very small in magnitude. This can be accomplished using the option `view=0..5` which restricts the plot range for the z-coordinate between 0 and 5, for example. Type this command after the plot range for x and y are specified, but before you close the parentheses on the `plot3d` command. Try the z range from 0 to 0.001.

a. What is the secret message?

b. What happens to the graph as x and y get very large in magnitude? Explain why.

No graphs need to be turned in for this question.

2. Plot the function $f(x, y) = y^3 - 12y + 2x^2 + 4x + 4$ using `plot3d`.

   a. Using your graph and using the command `contourplot`, locate specifically any extrema (max's, min's or saddles). You can try and use the commands `maximize` or `minimize` but remember that they don't always work and often depend heavily on the search range you specify. Give the coordinates $(x, y, f(x, y))$ for each extrema and label these points on a print out of a good 3d-graph. Later in the course, we will see how to find the extrema algebraically.

   b. Draw contour plots near each of the extrema. What do these contour plots have in common with those of the functions $h_1(x, y) = x^2 + y^2$ and $h_2(x, y) = x^2 - y^2$? How do you get the contour plot of the saddle to include the two crossing lines passing through the saddle point? Print out and turn in a contour plot for each of the extrema.

   c. Is there a global maximum or global minimum for this function? In other words, as (x, y) is varied over the whole xy-plane, is there a function value which is the least or most possible? What would you say the range of this function is? Explain your answers.

3. Plot the function $g(x, y) = x^2/5 + 2e^{-x^2/2-y^2/2} - 2\sin x\, e^{-x^2/4-y^2/4}$ which can be assigned to g by typing

```
g := (x,y) -> x^2/5 + 2*exp(-x^2/2 - y^2/2 ) -2*sin(x)*exp(-x^2/4 - y^2/4);
```

   a. Using your graph, the command `contourplot`, and any other commands you think helpful, locate as best as possible the extrema (there are a total of five). Give the coordinates $(x, y, g(x, y))$ for each point. No graphs are required to be turned in for this question.

   b. Is there any symmetry in this function which is demonstrated in the graphs? Where? How can you see this analytically from the formula for the function? How does it help you with your answers to part **a**?

   c. Is there a global maximum or global minimum for $g(x, y)$? If so, what are they? What would you say the range of this function is? Explain your answers.

4. (An interesting function) Investigate $h(x, y) = \sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}$. In Maple, the square root function syntax is `sqrt( )` where the radicand is placed inside the parentheses.

a. Does this function have any extrema? Specifically, are there points where the contour plots in a neighborhood of the point resemble those of a maximum, minimum or saddle? What are the contours or level curves? Be precise. The curves should look familiar. Print out and turn in a contour plot.

b. What does the cross-section of the 3-d graph in the xz-plane look like? Do some algebra here and sketch it in the xz-plane. How does this sketch help confirm the answer to part **a**?

c. Does $h(x, y)$ have a maximum or minimum value? If so, what are they? What would you say the range of this function is? Explain your answers.

d. This function has a rather nice geometric interpretation which helps shed light on its behavior. Just thinking in the xy-plane, use the distance formula to interpret the function $h(x, y)$. *Hint:* Replace y^2 with $(y-0)^2$ and plot three key points. What does the triangle inequality imply about the function $h(x, y)$? Once you have this interpretation, use it to explain your answers to parts **a, b** and **c**.

5. (An Ant's Path) Suppose you are an ant who wants to cross a rectangular plate ($0 \leq x \leq 6.3, -2 \leq y \leq 2$) whose temperature is given by the function $T(x, y) = (1 - y)\sin^2 x$. You start on the side $y = -2$ and want to get to the side $y = 2$ by avoiding the hottest points on the plate.

   a. What is the coolest point(s) on the side $y = -2$? Explain. Print out a contour plot of the temperature and draw a path which you would follow to get from the coolest point on $y = -2$ to the coolest point on $y = 2$. Recall that $\sin^2 x = (\sin x)^2$.

   b. What are the hottest point(s) on the side $y = -2$? On the same contour plot as in part **a**, draw a path from the hottest point on the side $y = -2$ to the coolest point on the side $y = 2$.

6. (When the Computer Fails Us) Consider the function $t(x, y) = \tan(2x - y)$.

   a. Without using the computer, sketch the contour plot of $t(x, y)$ in the plot range $-2 \leq x \leq 2, -2 \leq y \leq 2$. Be sure to use enough levels to get an accurate and informative sketch.

   b. Using Maple, enter the following command:

   ```
   contourplot(tan(2*x-y),x=-2..2,y=-2..2);
   ```

   What happens? Does the computer's picture agree with your picture from part **a**? Try increasing the grid size to get a finer plot. What happens? Is there any improvement?

   c. Now try specifying the contours for the computer by giving the option `contours=[-2,-1,0,1,2]`. How well does the computer's picture compare with yours? Where are there problems? Why do you think the computer is having trouble? Draw some conclusions about the usefulness of packages like Maple based on this example.