MATH 241 Exam #2 Solutions

November 7, 2001

Name: ANSWER KEY

Read all **DIRECTIONS** carefully. Please show all work using extra paper if necessary. There are a total of 100 points on the exam. Good luck.

1. Which vector is normal to the **surface** given by the graph of $z = f(x, y)$? (8 pts.)

   (a) ∇f

   (b) $f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

   (c) $f_x \vec{i} + f_y \vec{j} - \vec{k}$

   (d) $-f_y \vec{i} + f_x \vec{j}$

   Answer: (c) We must make the graph into a **level** surface of a 3-variable function by bringing z onto the other side, $f(x, y) - z = 0$. Letting $g(x, y, z) = f(x, y) - z$, we have that $g = 0$ is the level surface which is the graph of f. Then, $\nabla g = f_x \vec{i} + f_y \vec{j} - \vec{k}$ is perpendicular to the level surface.

2. If $z = ye^x$, $x = \ln(u^2 + v^2), y = -v^2$, use the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$. Simplify your answers. (12 pts.)

   Answer: Using the Chain Rule:

$$
\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= ye^x \cdot \frac{2u}{u^2 + v^2} + e^x \cdot 0 \\
&= -v^2 e^{\ln(u^2+v^2)} \cdot \frac{2u}{u^2 + v^2} \\
&= -2uv^2
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= ye^x \cdot \frac{2v}{u^2 + v^2} + e^x \cdot (-2v) \\
&= -v^2 e^{\ln(u^2+v^2)} \cdot \frac{2v}{u^2 + v^2} + e^{\ln(u^2+v^2)} \cdot (-2v) \\
&= -2v^3 - 2v(u^2 + v^2) \\
&= -4v^3 - 2u^2 v
\end{aligned}
$$

3. Let $f(x,y) = e^{(x^2-y^2)}$.

(a) Find the gradient of f, ∇f at the point $(1,-1)$.

Answer: $\nabla f(x,y) = 2xe^{x^2-y^2}\vec{i} - 2ye^{x^2-y^2}\vec{j}$. Evaluating at $(1,-1)$ gives $\nabla f(1,-1) = 2\vec{i} + 2\vec{j}$.

(b) Find the directional derivative of f at the point $(1,-1)$ in the direction of the vector $\vec{v} = -4\vec{i} + 3\vec{j}$.

Answer: First convert $\vec{v}$ into a unit vector by dividing by its length, $||\vec{v}|| = 5$, giving $\vec{U} = -\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$. The directional derivative, when f is differentiable, is given by

$$f_{\vec{U}} \;=\; \nabla f \cdot \vec{U} \;=\; (2\vec{i} + 2\vec{j}) \cdot (-\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}) \;=\; -\frac{2}{5}.$$

(c) Find the direction of greatest **decrease** for f at the point $(1,-1)$.

Answer: The direction of greatest decrease is in the direction of $-\nabla f$. We make it a unit vector by dividing by its length:

$$\frac{-\nabla f}{||-\nabla f||} \;=\; \frac{-2\vec{i} - 2\vec{j}}{2\sqrt{2}} \;=\; \frac{-1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

(d) Find the level curve through $(1,-1)$. Sketch the curve and the vector $\nabla f(1,-1)$ with its tail at $(1,-1)$. (20 pts.)

Answer: To find the level curve through $(1,-1)$ we evaluate $f(1,-1) = 1$ to see that we must find the points (x,y) which satisfy $e^{x^2-y^2} = 1$. This is equivalent to $x^2 - y^2 = 0$ (take natural log of both sides or think about the exponent). Solving $x^2 = y^2$ yields $y = \pm x$. Since we want the curve passing through $(1,-1)$ the answer is $y = -x$. Notice that the vector $\nabla f(1,-1) = 2\vec{i} + 2\vec{j}$ is perpendicular to this line.

4. Let $g(x, y) = \cos x \, \sin y$.

(a) Compute the quadratic Taylor approximation for $g(x, y)$ about the point $(0, \pi/2)$.

Answer: Note that $g(0, \pi/2) = 1$. We compute the first and second-order partials:

$$\begin{aligned} g_x &= -\sin x \sin y \\ g_y &= \cos x \cos y \\ g_{xx} &= -\cos x \sin y \\ g_{yy} &= -\cos x \sin y \\ g_{xy} &= -\sin x \cos y \end{aligned}$$

Evaluating at $(0, \pi/2)$ gives 0 except for $g_{xx}(0, \pi/2) = g_{yy}(0, \pi/2) = -1$. The quadratic approximation is therefore

$$Q(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}(y - \pi/2)^2.$$

(b) Using the information from part **(a)**, sketch the contour plot for g near the point $(0, \pi/2)$. (20 pts.)

Notice that since both g_x and g_y are 0 at $(0, \pi/2)$, this point is a critical point. It is a local maximum by the second-derivative test or by visualizing the graph of $Q(x, y)$ as a bowl opening downward. Thus we expect to see circles centered around $(0, \pi/2)$ which get closer together as you move away from the critical point.

Alternatively, we can approximate the contour plot by solving

$$c = 1 - \frac{1}{2}x^2 - \frac{1}{2}(y - \pi/2)^2$$

or

$$x^2 + (y - \pi/2)^2 = -2(c - 1).$$

The contours are therefore circles of radius $\sqrt{2 - 2c}$ centered at $(0, \pi/2)$.

5. **(a)** Suppose that $f(x, y)$ is differentiable at (a, b), with linear approximation given by $L(x, y)$. Give a qualitative, geometric description of what this means in one or two sentences. (6 pts.)

   Answer: The graph of $L(x, y)$ is the tangent plane to the graph of $f(x, y)$. As we zoom in on the graph of f at the point (a, b), the graph resembles this tangent plane.

   (b) Suppose that $f(x, y)$ is differentiable at (a, b), with linear approximation given by $L(x, y)$. Give the algebraic, limit definition of what this means. (6 pts.)

   Answer: Let $E(x, y) = f(x, y) - L(x, y)$ be the error between the function and the linear approximation. We say that $f(x, y)$ is differentiable at (a, b) iff

$$\lim_{(h,k)\to(0,0)} \frac{E(a + h, b + k)}{\sqrt{h^2 + k^2}} = 0.$$

   (c) Give an example of a function $f(x, y)$ which is continuous at $(0, 0)$ but not differentiable at $(0, 0)$. (6 pts.)

   Answer: The cone given by $f(x, y) = \sqrt{x^2 + y^2}$ is a good example. The example from the second computer project, problem 2, also works. Other solutions include $f(x, y) = |x|$ or $f(x, y) = |y|$.

6. Let $f(x,y) = xy - x^4 - \frac{1}{2}y^2$.

(a) Find all the critical points of $f(x,y)$.

Answer: We compute $f_x = y - 4x^3$ and $f_y = x - y$. The critical points are found by solving the system of equations

$$\begin{aligned} y - 4x^3 &= 0 \\ x - y &= 0 \end{aligned}$$

The second equation implies $y = x$ which yields after substitution $4x^3 - x = 0$. This has **three** solutions after factoring, $x = 0, \pm 1/2$. Since $y = x$, we substitute each x-value into this simple equation to get the three critical points $(0,0), (1/2, 1/2), (-1/2, -1/2)$.

(b) Use the second-derivative test to classify each critical point as either a maximum, minimum or saddle point.

Answer: The second partials are $f_{xx} = -12x^2, f_{yy} = -1, f_{xy} = 1$, so the discriminant is given by $D = 12x^2 - 1$. For $x = 0$, we obtain $D = -1 < 0$, so that $(0,0)$ is a saddle point. For both $x = \pm 1/2$, we have $D = 2 > 0$ and $f_{xx} = -3 < 0$, which means $(1/2, 1/2)$ and $(-1/2, -1/2)$ are relative maximums.

(c) Does $f(x,y)$ have a global max or a global min? Explain. (22 pts.)

Answer: There is no global min since for a fixed y-value (say $y = 0$), $f(x,0) = -x^4$ approaches $-\infty$ as x gets larger in absolute value. The global max occurs at $(1/2, 1/2)$ and $(-1/2, -1/2)$ with a value of $f(1/2, 1/2) = f(-1/2, -1/2) = 1/16$. There can be no higher finite value because it would have to be a critical point and we have found all of the critical points. The function can not approach $+\infty$ because the x^4 term dominates and has a negative coefficient.