MATH 241 Exam #1

October 3, 2001

Name: SOLUTIONS

Read all **DIRECTIONS** carefully. Please show all work using extra paper if necessary. There are a total of 100 points on the exam.

1. Multiple Choice: There is exactly one answer to each question. (16 pts.)

(i) Which equation gives an elliptic paraboloid (bowl) with vertex at the origin and opening outwards along the positive y-axis?
(A) $z^2 = 1 - x^2 - y^2$
(B) $x = y^2 + z^2$
(C) $y = x^2 + z^2$
(D) $z = x^2 - y^2$
(E) $z^2 = x^2 + y^2$
Ans: (C) Choose the equation which has circles for cross-sections parallel to the xz-plane.

(ii) Which vector is perpendicular to $\vec{v} = 3\vec{i} - 5\vec{j} + \vec{k}$?
(A) $\vec{w} = \vec{i} + \vec{j}$
(B) $\vec{w} = -3\vec{i} + \vec{k}$
(C) $\vec{w} = -\vec{i} + 3\vec{k}$
(D) $\vec{w} = -2\vec{i}$
Ans: (C) Choose the vector whose dot product with $\vec{v}$ is zero.

(iii) $\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$, $x \neq y$
(A) $= 0$
(B) $= 1$
(C) $= \infty$
(D) does not exist
(E) not enough information given to do the problem
Ans: (D) Taking $y = mx$ gives the expression $(m+1)/(m-1)$ which varies with m.

(iv) The contours of the function $f(x, y) = x^2 - y + 2$ are
(A) ellipses
(B) parabolas
(C) hyperbolas
(D) parallel lines
(E) do not exist
Ans: (B) Setting $f(x, y) = c$ gives a family of parabolas.

2. Find an equation for a level surface of $f(x,y,z) = \cos\sqrt{x^2+y^2+z^2}$ for the function value $f = -1$. Describe and sketch the surface in xyz-space. How many actual level surfaces exist corresponding to the value $f = -1$? Explain. (15 pts.)

Ans: Setting $f(x,y,z) = -1$ gives the equation

$$\cos\sqrt{x^2+y^2+z^2} = -1$$

which, taking the inverse cosine of both sides, yields

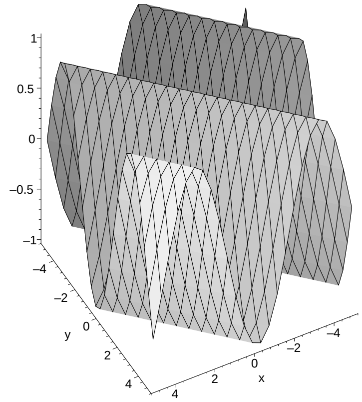
$$\sqrt{x^2+y^2+z^2} = \pi$$

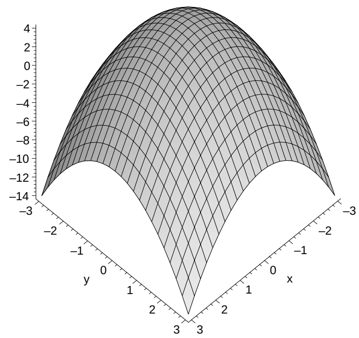
or $x^2+y^2+z^2 = \pi^2$. However, note that because the cosine function is periodic with period 2π, there are infinitely many choices for the inverse cosine of -1. (As a function, we define $\cos^{-1} x$ to only take values between 0 and π, but when solving an equation, we must search for **all** solutions.) Therefore, there are an infinite number of level surfaces of the form

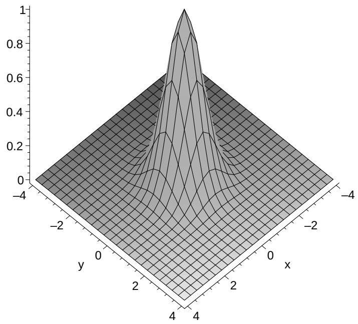
$$x^2+y^2+z^2 = k^2\pi^2, \quad \text{where } k \text{ is an odd positive integer}$$

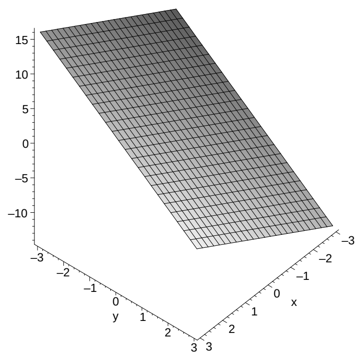
These are spheres centered at the origin of radius $k\pi$, where k is an odd positive integer.

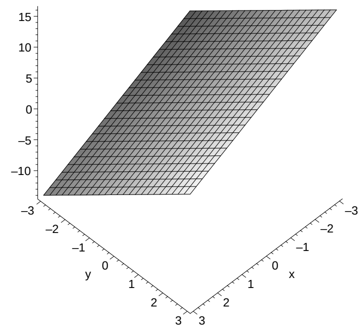
3. Match the graph with the correct function. No explanation required. (15 pts.)











(i) $z = 4 - x^2 - y^2$ <u>B</u>

(ii) $z = 2x - 3y + 1$ <u>D</u>

(iii) $z = \sin(x + y)$ <u>A</u>

(iv) $z = e^{-x^2-y^2}$ <u>C</u>

(v) $z = -2x + 3y + 1$ <u>E</u>

4. Given the vectors $\vec{v} = \vec{i} + 7\vec{k}$ and $\vec{w} = 2\vec{i} - 4\vec{j} + 4\vec{k}$. (18 pts.)

(i) Compute $\vec{v} \cdot \vec{w}$.

(ii) Find the angle between $\vec{v}$ and $\vec{w}$.

(iii) Find a vector perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$.

(iv) Set up an expression for the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$.

(i) $\vec{v} \cdot \vec{w} = (2)(1) + (0)(-4) + (7)(4) = 30$ Remember that the dot product is a number!

(ii) Using the formula $\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos\theta$ we obtain

$$30 = \sqrt{50}\sqrt{36}\cos\theta = 30\sqrt{2}\cos\theta$$

so that $\cos\theta = 1/\sqrt{2}$. Therefore, the angle between $\vec{v}$ and $\vec{w}$ is $\pi/4$.

(iii) By definition, the cross product of $\vec{v}$ and $\vec{w}$ is perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$. We compute

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 7 \\ 2 & -4 & 4 \end{vmatrix} = \vec{i}(0 + 28) - \vec{j}(4 - 14) + \vec{k}(-4 - 0)$$

so our answer is $28\vec{i} + 10\vec{j} - 4\vec{k}$.

(iv) The area of the parallelogram formed by $\vec{v}$ and $\vec{w}$ is given by $||\vec{v} \times \vec{w}||$, that is to say, find the magnitude of the vector in part (iii)

$$||\vec{v} \times \vec{w}|| = \sqrt{28^2 + 10^2 + (-4)^2} = 30.$$

Note that since we know the angel between $\vec{v}$ and $\vec{w}$ is $\pi/4$ we could also use the formula

$$||\vec{v} \times \vec{w}|| = ||\vec{v}||||\vec{w}|| \sin\theta = \sqrt{50}\sqrt{36}/\sqrt{2} = 30.$$

5. Fill in the blanks and then give an expression for the linear function $L(x,y)$. Be sure to show your work. (16 pts.)

x/y	4	6
1	9	5
2	12	8
3	15	11

The slope in the x-direction is given by

$$m = \frac{\Delta z}{\Delta x} = \frac{15-12}{3-2} = 3$$

and the slope in the y-direction is given by

$$n = \frac{\Delta z}{\Delta y} = \frac{8-12}{6-4} = -2.$$

You can use these values to fill out the table.

Then the equation of the linear function is given by $L(x,y) = mx + ny + c = 3x - 2y + c$. Using $L(1,4) = 9$ as a sample point, we obtain

$$9 = 3(1) - 2(4) + c \implies c = 14$$

so

$$L(x,y) = 3x - 2y + 14$$

is the linear function we seek.

6. Two women plan to row across a river from South to North. They are skilled rowers and can row at a rate of 12 mph. The current of the river is flowing Eastward at a rate of 3 mph. There is a light wind blowing at a rate of 5 mph out of the southwest at an angle of $\theta = \sin^{-1}(4/5)$ south of west. (See figure.) (20 pts.)

(i) Give the vectors corresponding to the current and the wind.

(ii) In what direction should they row so there boat heads due North?

(iii) Assuming they are rowing due North, will they get across the river faster than if there was no current and no wind? In other words, do the wind and current help their progress or impede it? Explain.

(i) The current is given by the vector $\vec{c} = 3\vec{\imath}$. To find the vector for the wind we need to find the unit vector representing its direction. Since $\theta = \sin^{-1}(4/5)$ we know that $\sin\theta = 4/5$. Using right-triangle trig., we find $\cos\theta = 3/5$. (Its a 3-4-5 right triangle.) Since the vector is directed towards the first quadrant, we see that the wind $\vec{w}$ is given by

$$\vec{w} = 5(\frac{3}{5}\vec{\imath} + \frac{4}{5}\vec{\jmath}) = 3\vec{\imath} + 4\vec{\jmath}.$$

(ii) Summing the two vectors $\vec{c}$ and $\vec{w}$, we see that the total force on the boat is given by $6\vec{\imath} + 4\vec{\jmath}$. The direction they row in, taken together with their speed of 12, should result in an $\vec{\imath}$ component of -6 so as to cancel the Eastward force of the wind and current combined. Letting $\vec{v}$ be the velocity vector of the boat, we can write $\vec{v} = 12(\cos\beta + \sin\beta)$. Then, we need to solve the equation

$$12\cos\beta = -6$$

which gives $\cos\beta = -1/2$ and thus $\beta = 2\pi/3$. They must row 60 degrees North of West.

(iii) Assuming that they are rowing 60 degrees North of West, their velocity vector is $\vec{v} = -6\vec{\imath} + 12\,\sin(2\pi/3) = -6\vec{\imath} + 6\sqrt{3}\vec{\jmath}$. Taken together with the force of the wind and current, we see that their overall velocity with respect to the bank of the river is given by

$$\vec{v} + \vec{w} + \vec{c} = (4 + 6\sqrt{3})\vec{\jmath}.$$

Since $4 + 6\sqrt{3} > 12$, the women are actually helped along by the force from the wind. (The current does not help them because it has no $\vec{\jmath}$ component.) Their overall speed is approximately 14.392 mph.

Note that the speed of the current certainly effects the answer. For example, if the current were at 9 mph, then the women would have to row due West just to cancel out the force from the wind and current in the $\vec{\imath}$ direction. However, this would mean their resulting velocity vector with respect to the river bank would only be $4\vec{\jmath}$ and they would travel at a rate much slower than if their was no current or wind.