Multivariable Calculus: Sept. 28, 2001

Worksheet on Tangent Planes

The focus of this worksheet is to use Maple to see how the tangent plane to a differentiable function is the best linear approximation to the function. Recall from Calc 1 that the tangent line to a function gives the best linear approximation to the graph. In fact, as you zoom in on the curve at a particular point P, the curve and tangent line become indistinguishable. This is easy to demonstrate using Maple.

A Calc 1 Example

Suppose that $f(x) = \sin x$. Calculate the equation of the tangent line to the graph when $x = \pi/3$. Do this without the computer. (Review your trig!) We want to plot this line, which I will call, $y = mx + b$ (you have found m and b) along with the graph of $\sin x$. To do this you type:

```
plot([m*x + b,sin(x)],x=0..Pi, color=[red,blue]);
```

This should draw the curve in blue and the tangent line in red. Make sure your tangent line is actually tangent to the graph at $x = \pi/3$. Recall that `Pi` is how you type in π and `sqrt(x)` is how you get $\sqrt{x}$. Next, we want to zoom in on the graph near $x = \pi/3$. To do this without retyping the `plot` command over and over again, we define a zooming command by typing the following:

```
zoomplot:= h -> plot([m*x + b,sin(x)], x=Pi/3-h..Pi/3+h, color=[red,blue]):
```

This creates the command `zoomplot(h)` which you can now use at will to zoom in on the graph near $x = \pi/3$. For example, if I want the plot range to be centered about $\pi/3$ with a distance of 0.3 to either side, I type `zoomplot(0.3)`. This should give you the graph and the tangent line drawn on the x range of $(\pi/3 - 0.3, \pi/3 + 0.3)$. Use the `zoomplot` command to keep zooming in on your graph until you see that the tangent line and curve become nearly identical.

Two important notes: First, you can define a constant to a letter by typing `b := Pi/10 - 3.6739;` for example. This way you can refer to the constant as b each time rather than having to type it in over and over again. Second, your zoomplot command only works for the example given ($\sin x$). To do a different function and tangent line and point $x = a$, you must return to the definition of `zoomplot` and type in the new functions you want.

Tangent Planes

Now we will create the equivalent approximation but to a function of two variables. Consider the function $f(x, y) = 1 - x^2 - y^2$. Plot the function over the range $-4 < x < 4, -4 < y < 4$. We would like to zoom in on the function around the point $(-2, 1)$. As before, we can define a zooming function which will do this for us by typing:

```
zoomplot3d:= h -> plot3d(1-x^2-y^2, x=-2-h..-2+h,y=1-h..1+h,axes=framed):
```

If I want to draw the graph on the range $-3 < x < -1, 0 < y < 2$, for example, I type `zoomplot3d(1)` giving one units length to either side of $x = -2$ and $y = 1$. Zoom in on the graph near $(-2, 1)$ using the `zoomplot3d(h)` command. What do you notice?

Now plot the linear function $L(x, y) = 4x - 2y + 6$ over the range $-3 < x < -1, 0 < y < 2$. Compare with your zooms of the function $f(x, y)$. This plane is called the **tangent plane** to the function $f(x, y)$ at the point $(-2, 1)$ and is the analog of the tangent line to a one-variable function at a point $x = a$.

Just as in Calc 1, we use partial derivatives to find the particular slopes of the tangent plane. Recall that $L(x, y) = mx + ny + c$ gave the equation of a plane, where m is the slope in the x-direction, n gives the slope in the y-direction and c is the z-intercept. It shouldn't be a big surprise to discover that when looking for the equation of a tangent plane to a function $f(x, y)$, m is obtained by computing the partial derivative of f with respect to x. After all, this tells us the slope of the graph in the x-direction for fixed y. Likewise, n is obtained by computing the partial derivative of f with respect to y. This tells us the slope of the graph in the y-direction for fixed x.

To obtain m in our example, compute $f_x(-2,1)$, the partial derivative of the function with respect to x evaluated at the point $(-2,1)$. Notice that you get $m = f_x(-2,1) = 4$. Similarly, to obtain n for our example, compute $f_y(-2,1)$, the partial derivative of f with respect to y evaluated at the point $(-2,1)$. Here we obtain $n = f_y(-2,1) = -2$. Finally, the tangent plane should pass through the point on the graph $(-2,1,f(-2,1)) = (-2,1,-4)$. Use this point to find the z-intercept by plugging it into $z = 4x - 2y + c$ and solving for c. This gives $c = 6$. Thus, the equation of the tangent plane to $f(x, y) = 1 - x^2 - y^2$ at the point $(-2,1)$ is given by

$$z = 4x - 2y + 6.$$

Of course you are probably anxious to see the graph and the tangent plane together. You can achieve this by typing:

```
plot3d({4*x-2*y+6,1-x^2-y^2},x=-4..4,y=-4..4,axes=framed);
```

I do not know how to make them different colors (maybe you can figure that out) but it didn't seem as critical here because of the relative size of the two surfaces. Click on your graph and rotate around to see where the two surfaces touch. Zoom in on the point $(-2,1)$ to see the surfaces becoming more and more the same.

In sum, the equation of the tangent plane for $f(x, y)$ at the point (a, b) is given by

$$z = f_x(a,b)\,x + f_y(a,b)\,y + c$$

where c is determined by making sure the plane passes through the point $(a, b, f(a, b))$. Another formula often used is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

which just gives the specific value of c.

Exercises:

1. For the function $f(x, y) = 1 - x^2 - y^2$, compute the tangent plane at the point $(0,0)$. What is significant about this point? Plot the tangent plane and the function on the same graph.

2. For the function $g(x, y) = y^3 - 12y + 2x^2 + 4x + 4$, compute the tangent plane at the point $(-1,2)$. What is significant about this point? Plot the tangent plane and the function on the same graph. Compare with the previous problem.

3. For the function $h(x, y) = \sin(xy)$, compute the tangent plane at the point $(0,0)$. What kind of point is $(0,0)$? Plot the tangent plane and the function on the same graph, in an appropriate plot range.

4. Based on Questions 1–3, what can you conclude about the tangent plane to a maximum, minimum or saddle? How is this similar to what we know from Calc 1 when considering the tangent line at an extremum? Given a function $f(x, y)$ in general, how do we locate the extrema? Compare this with how we find extrema for one-variable functions.

5. Define a Maple function called `tanplane(a,b,h)` which plots both the function $f(x, y) = 1 - x^2 - y^2$ and its tangent plane at the point (a, b) over the plot range $a - h < x < a + h, b - h < y < b + h$. For example, when you type `tanplane(-1,1,2)`, you should get a 3d plot of $f(x, y)$ and its tangent plane at the point $(-1, 1)$ drawn over the plot range of $-3 < x < 1, -1 < y < 3$. The first part of your command should look like

   ```
   tanplane := (a,b,h) -> plot3d({ ...  );
   ```

   You must fill in the important information inside the `plot3d` command. Once you get your command working, use it to view various tangent planes on the graph of $f(x, y)$ to get a better understanding of the function.

6. Repeat Question 5 for the function $h(x, y) = \sin(xy)$.