Multivariable Calculus: Oct. 15, 2001
Tangent Planes To Surfaces : A New Theorem?

1 Surfaces

Throughout the semester we have studied various surfaces in 3-space, eg. spheres, ellipsoids, bowls, hyperboloids, cones, planes, etc. Loosely defined, a **surface** is an object in space having the property that any small patch resembles a piece of a plane. For example, even though the Earth is spherical, if you stand at a point on the surface of the Earth, it looks flat. (Well most places anyway.) Alternatively, one can take a flat plane and wrap it up to form a sphere. We run into problems with a cone at its vertex, but everywhere else this definition makes sense. The surfaces we have been studying have equations like:

$$\begin{aligned} x^2 + 2y^2 + 3z^2 = 12 &\quad \text{ellipsoid} \\ x^2 + y^2 = 3z^2 &\quad \text{cone} \\ z = \sin(x+y) &\quad \text{river?} \\ 3x - 2y + 4z = 7 &\quad \text{plane} \end{aligned}$$

It is important to realize that **not** all surfaces are graphs of two variable functions. For example, an ellipsoid is never the graph of a 2-variable function because each pair (x, y) has two z-values which can accompany it. Both $(1, 2, -1)$ and $(1, 2, 1)$ are on the ellipsoid $x^2 + 2y^2 + 3z^2 = 12$. This cannot possibly represent a function of any two variables because to solve for one of the variables involves the square root and thus two possible outcomes. Note that $y = x^2 + z^2$ can be thought of as a function of the variables x and z with y as the output variable. However, it is not a function of x and y because of the z^2 term. It also fails the vertical line test.

We must distinguish between two types of surfaces, a major source of confusion for many students. These are:

Level Surface The set of all points (x, y, z) which satisfy the equation $g(x, y, z) = s$ for some constant s where g is any function of **3** variables.

Graph of a two-variable function The set of all points (x, y, z) which satisfy the equation $z = f(x, y)$ where f is any function of **2** variables.

2 Tangent Planes to Surfaces

In Section 3.3 of the text, we learned how to find the tangent plane to the graph $z = f(x, y)$ at the point $(a, b, f(a, b))$. Using the linear approximation of $f(x, y)$ about (a, b) gives the equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The right-hand side of the previous expression is exactly the linear approximation $L(x, y)$ to f. Its **graph** is the tangent plane to the surface $z = f(x, y)$. This should make sense. If $L(x, y)$ approximates the function $f(x, y)$, then the graph of $L(x, y)$ should approximate the graph of $f(x, y)$. From a previous worksheet, we know that as you zoom in on a surface, it looks more and more like the tangent plane at that point. Therefore, the graph of $L(x, y)$ is exactly the tangent plane to the surface $z = f(x, y)$.

We run into problems when we consider level surfaces. Suppose we wish to find the tangent plane to the level surface $g(x, y, z) = s$ at the point (a, b, c). Note that $g(a, b, c) = s$, otherwise the point would not be on the surface and it would be a moot point. (That's a pun, get it.) Next, consider the linear approximation of $g(x, y, z)$ about (a, b, c),

$$L(x, y, z) = g(a, b, c) + g_x(a, b, c)(x - a) + g_y(a, b, c)(y - b) + g_z(a, b, c)(z - c).$$

The graph of this function, although linear, exists in 4 dimensions! The output variable, say w, adds another dimension. Its graph would be "flat" in 4-space and it would certainly approximate the graph of $g(x, y, z)$, but it would be about the point (a, b, c, s) in 4-space and not the tangent plane that we seek. We don't want the tangent "thing" to the graph in 4-space, we want the tangent plane to the level surface in 3-space. So how do we find the equation of the tangent plane to a level surface?

Theorem 2.1 *(Andrade, Christo, Hardy, et. al., 2001) Suppose that $g(x, y, z)$ is a differentiable function. The tangent plane to the level surface $g(x, y, z) = s$ at the point (a, b, c) is given by*

$$g_x(a, b, c)\,(x - a) \;+\; g_y(a, b, c)\,(y - b) \;+\; g_z(a, b, c)\,(z - c) \;=\; 0. \tag{1}$$

We will give two proofs, one using the gradient of g, the other using cross sections and your imagination.

Proof 1: One of the key properties of ∇g is that it is always perpendicular to level surfaces $g = s$ (level curves if g is a two-variable function). Therefore, the vector $\vec{n} = g_x(a, b, c)\,\vec{i} + g_y(a, b, c)\,\vec{j} + g_z(a, b, c)\,\vec{k}$ is perpendicular to the level surface $g(x, y, z) = s$ through the point (a, b, c). Since it is perpendicular to the surface, it is the normal vector to the tangent plane we seek. If (x, y, z) represents any point on the tangent plane, then the vector $\vec{v} = (x - a)\,\vec{i} + (y - b)\,\vec{j} + (z - c)\,\vec{k}$ lies in the plane and must be perpendicular to the normal vector $\vec{n}$. Setting the dot product of $\vec{n}$ and $\vec{v}$ equal to zero gives equation (1). Since this is true for any point (x, y, z) in the plane, we have found the defining equation of the tangent plane. QED.

Proof 2: This proof involves cross-sections and is more ingenious. Consider the linearization of $g(x, y, z)$ about (a, b, c),

$$w = g(a, b, c) + g_x(a, b, c)(x - a) + g_y(a, b, c)(y - b) + g_z(a, b, c)(z - c). \tag{2}$$

The graph of this function, although in $xyzw$-space, is still a "tangent" approximation to the graph of $w = g(x, y, z)$ in $xyzw$-space. We may not be able to visualize it in 4 dimensions, but we can take a cross-section! The obvious cross-section to take is $w = s$. The cross-section of the graph of the function g when $w = s$ is the level surface we are interested in $g(x, y, z) = s$. Similarly, the cross-section of the graph of the "tangent" approximation in equation (2) is exactly the tangent plane we are after. Since it is the best linear approximation to the 4-d graph of g, its cross-section is the best linear approximation to the cross-section $g = s$, ie. it's the tangent plane to our level surface! Simply setting $w = s$ in equation (2) and recalling that $g(a, b, c) = s$ quickly gives formula (1). QED.

A Remark: This cross-section argument works in lower dimensions as well. For example, consider the graph of the function $f(x, y) = x^2 + y^2$, our old friend the elliptic paraboloid. Suppose you wanted the equation of the tangent **line** to the level **curve** $f(x, y) = x^2 + y^2 = 1$ at the point $(a, b) = (3/5, 4/5)$. So we are in the xy-plane now, not xyz-space. The formula to calculate this is simply

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0$$

which in this case yields, $6x+8y=10$. (Note how much easier this method is than Calc 1 methods.) This formula can be proven using either of the methods of proof given above. For the cross-section method, just take the linearization of $f(x,y)$ about (a,b) and set $z=s=f(a,b)$. Geometrically, the cross-section of the bowl and its tangent plane gives a circle and its tangent line!

3 Treating the graph $z=f(x,y)$ as a level surface

There is a simple trick to treating the graph of a two variable function $f(x,y)$ as a level surface of a three-variable function $g(x,y,z)$. Simply take the equation $z=f(x,y)$ and subtract z from both sides to get $f(x,y)-z=0$. Now define $g(x,y,z)$ to be the function

$$g(x,y,z) \;=\; f(x,y)-z.$$

The level surface $g(x,y,z)=0$ is exactly the graph of $f(x,y)$ in xyz-space. Now we can apply Theorem 2.1 to the function g to find the tangent plane to f at a given point $(a,b,f(a,b))$. Sometimes adding an extra dimension simplifies a problem.

Here is a simple example done using both methods. Let $f(x,y)=xy^2-e^{xy}$. Find the tangent plane to the graph $z=f(x,y)$ at the point $(0,2)$.

Solution 1 (Treating the graph as a level surface) First note that $f(0,2)=-1$ so up on the graph, the point of tangency is $(0,2,-1)$. Set $g(x,y,z)=f(x,y)-z=xy^2-e^{xy}-z$ and compute the partial derivatives of g:

$$\begin{aligned} g_x &= y^2-ye^{xy} \\ g_y &= 2xy-xe^{xy} \\ g_z &= -1 \end{aligned}$$

Notice that $g_x=f_x$ and $g_y=f_y$ which will always occur if $g(x,y,z)=f(x,y)-z$. Evaluating each partial at $(0,2,-1)$ gives the normal vector to the tangent plane, $\vec{n}=2\vec{\imath}-\vec{k}$. Now use the formula for a plane $ax+by+cz=d$ to obtain $2x-z=d$ and plug in the point $(0,2,-1)$ to see that $d=1$. Alternatively, you can use formula (1) to get $2(x-0)+0(y-2)-(z+1)=0$ or $2x-z=1$.

Solution 2 (Linear approximation) The graph of the linear approximation to $f(x,y)$ is exactly the tangent plane. It is found using the formula

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b).$$

The partial derivatives f_x and f_y are exactly the same as g_x and g_y above. So $f_x(0,2)=2$ and $f_y(0,2)=0$. Plugging this into the formula above and letting $z=L(x,y)$ gives

$$z=-1+2(x-0)+0(y-2)$$

or $2x-z=1$.

Try an example: Let $f(x,y)=\cos x\sin y$ and let S be the surface $z=f(x,y)$.

a) Find a normal vector to the surface S at the point $(0,\pi/3,\sqrt{3}/2)$.

b) What is the equation of the tangent plane to the surface S at the point $(0,\pi/3,\sqrt{3}/2)$?