Multivariable Calculus: Sept. 6, 2001

Worksheet on Visualizing Functions of 2 Variables

We have been studying how to visualize graphically functions of two variables, $f(x, y)$. The first method, drawing a graph in three dimensions, treats the dependent variable z (the output of the function) as the height above the input variables x and y. If the point $(2, 4, 5)$ is on the graph, then we know that $f(2, 4) = 5$. This is useful in that we can obtain general information about the function — is it increasing or decreasing or does it have a relative maximum or minimum? However, the graphs are often hard to understand (or too complicated to draw) and hard to obtain specific numerical values from.

Enter the second method of interpreting $f(x, y)$: the **contour plot**. The contour plot is a 2-d visualization of the function obtained by drawing the curves in the xy-plane corresponding to a *fixed* function value or height. These curves are called **level curves** or **contours**. (Think of a topographical map.) Here we fix the output value $z = c$ and sketch the curve in the xy-plane whose function values are all c. This is equivalent to taking a cross-section $z = c$ in our 3-d graph and then projecting the image down onto the xy-plane. As we vary c we obtain different contours, but we sketch them all on the same 2-d graph in the xy-plane. This yields the contour plot of the function.

To investigate one of the examples we have done in class (my canonical example), type the following:

```
plot3d(x^2+y^2,x=-3..3,y=-3..3,axes=framed);
```

This should give you a graph of the function $f(x, y) = x^2 + y^2$. The `axes=framed` command displays the axes to give you a frame of reference. The coordinate ranges for each variable are separated by two periods. By clicking on your graph, you can adjust the frame of reference for your plot to get different views by dragging the mouse. This is very useful! Draw a plot of $f(x, y) = x^2-y^2$ (my favorite function) and vary the frame of reference to get a good image of the saddle (or potato chip). By the way, you can assign a letter to your function (so you don't have to keep typing it in) by typing

```
f:=(x,y)-> x^2 - y^2;
```

which, for example, assigns f to $x^2 - y^2$. Every time you want to refer to this function, you can just type `f(x,y)`. Another useful thing to know is how to get help in MAPLE. To get help on the command `plot3d` for instance, you type `?plot3d`. This opens up a help window with lots of information on the particular command.

To utilize some of the commands helpful in visualizing graphs of two variables we need to load the package "plots" by typing

```
with(plots):
```

Graph the contours of $f(x, y) = x^2 + y^2$ by typing

```
contourplot(x^2 + y^2, x=-3..3, y=-3..3);
```

By default, MAPLE plots the contours for 8 different values of c (the function value) which are equally spaced. As we discussed in class, this does NOT mean that the contours themselves will be equally spaced! You can add more contours by inserting the command `contours = 15` or by specifying them precisely in a list such as `contours=[0,0.5,1,1.5,2,2.5,3,3.5,4]`. In order to plot the contours, MAPLE samples the values of the function on a grid consisting of 25 points in the x-direction and 25 points in the y-direction. You can increase this amount (to get smoother contours) by typing `grid=[35,35]` for example. Try executing the following command:

```
contourplot(x^2+y^2,x=-3..3,y=-3..3,contours=[0,0.5,1,1.5,2,2.5,3,3.5,4],grid=[35,35]);
```

Two other useful commands for viewing contours are the commands `densityplot` and `contourplot3d`. The first command yields a shaded plot of the domain of the function in the xy-plane where the shading corresponds to the function value. Lighter shades mean larger function values while darker shades correspond to smaller function values. Check out the density plot of our saddle by typing

```
densityplot(x^2-y^2,x=-3..3,y=-3..3,grid=[50,50]);
```

The second command is a version of the `plot3d` command and shows horizontal slices of the graph. It displays 15 slices by default. You can increase or decrease the number of slices by using the same options as with the `contourplot` command. Try the following to see the contours on the saddle in 3d:

```
contourplot3d(x^2-y^2,x=-3..3,y=-3..3,contours=20);
```

Some Exercises:

1. Investigate the function $f(x,y) = 3e^{(-(x+1)^2-y^2)} + 2e^{(-(x-1)^2-y^2)} - 2e^{(-x^2-(y-1)^2)}$ using the `plot3d` and `contourplot` commands. You can type in this function as

   ```
   f := (x,y) -> 3*exp(-(x+1)^2 - y^2) + 2*exp(-(x-1)^2 - y^2)
   - 2*exp(-x^2 - (y-1)^2);
   ```

   Find a good domain to view the function. What happens to the function as x and y get large? Why? Find the extrema (max's and min's) of f? Where do they occur? Are there any saddle points? If so, where? What do the contours look like near a saddle point? What do the contours look like near max's or min's? Find good domains to view the contours near these key points. To get specific values of the function you can type `evalf(f(a,b));` which gives a numerical value of the function at the point (a,b).

2. Investigate the function $g(x,y) = \sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}$ answering the same questions as above. This function is typed in using

   ```
   g := (x,y) -> sqrt((x+1)^2 + y^2) - sqrt((x-1)^2 + y^2);
   ```

   What is different about this function? Does it have extrema points? What do the contours look like? What symmetry do you notice? Explain the cross-section of the graph in the xz-plane. Why is this significant? Do some algebra here.