MATH 136-02, MATH 136-03 Calculus 2, Fall 2018

Computer Lab #2: Numerical Integration

DUE DATE: Friday, November 9, start of class

The goal for this lab project is to use Maple to compute, compare, and visualize different numerical techniques for approximating definite integrals. The algorithms we have been studying are left-hand and right-hand sums, the midpoint rule, the trapezoid rule, and Simpson's rule. In this lab you will compute the errors of each method and determine the rate of improvement as the number of subintervals is increased. Ultimately, you will identify which numerical integration technique is the most accurate and by what factor. This lab is intended to explore and supplement the material in Section 7.9 of the course text.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced. Please turn in one report per group, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully, writing in complete sentences. You should **type** your answers, leaving space for mathematical calculations or graphs. You may include calculations and graphs in an appendix at the end of your report. There are only **two graphs** to be turned in for this report.

Getting Started: Useful Maple Integration Commands

To execute some of the commands needed for this lab, you must load the following three packages in Maple. Typing

```
with(student):  with(plots):  with(Student[Calculus1]):
```

will load the pertinent packages. These packages contain commands for doing numerical integration as well as the animation routines you will be running. Don't forget to load these packages **every time** you begin a new Maple session.

Recall that left-hand and right-hand sums, the midpoint rule, the trapezoid rule, and Simpson's rule are methods for approximating the signed area under the graph of $f(x)$ over an interval $[a, b]$. We first divide the interval $[a, b]$ into n equal pieces (called a partition), yielding n subintervals of width $\Delta x = (b - a)/n$. Then, we construct rectangles (or trapezoids) over each subinterval by choosing a representative height determined by the function. For a left-hand sum, we evaluate the function at the left endpoints of each subinterval, while for a right-hand sum, we use the right endpoints. For the midpoint rule we evaluate the function at the midpoint of each subinterval, while the trapezoid rule averages the left and right-hand sums. (Recall from class that this is identical to finding the area under the trapezoid between the endpoints—hence the name.) Simpson's rule is a weighted estimate of the midpoint rule (2/3) and the trapezoid rule (1/3).

In this lab, we will investigate the accuracy of a given numerical integration technique and compare methods to each other, drawing some important conclusions. We define the error of an approximation to be

$$\text{Error} = |\text{Actual value} - \text{Approximation}|.$$

We begin with a left-hand sum. Typing

```
n := 10:
leftbox(exp(x^2),x=0..1,n,shading = yellow);
```

will draw 10 rectangles in yellow for a left-hand sum of $f(x) = e^{x^2}$ over $0 \leq x \leq 1$. Try it. You can also enter the function e^{x^2} by choosing e^a from the "Expression" drop down menu. Change n to $n = 50$ and draw 50 rectangles for the same function and same interval.

Typing `rightbox(exp(x^2),x=0..1,n,shading = yellow);` draws the right-hand sum over the same interval with n rectangles, while `middlebox(exp(x^2),x=0..1,n,shading = yellow);` produces a plot of the midpoint rule. Notice that in this case, part of each rectangle is above the curve and part is below, leading to a better approximation than left-hand or right-hand sums.

It would be nice to compare the left-hand, right-hand, and midpoint sums visually by placing them side by side each other. This is easily done using the `display` and `array` commands. To display the three sums described above side by side, type the following:

```
n := 10:
Lplot:=leftbox(exp(x^2),x=0..1,n,shading=yellow,title="Left-hand Sum"):
Rplot:=rightbox(exp(x^2),x=0..1,n,shading=yellow,title="Right-hand Sum"):
Mplot:=middlebox(exp(x^2),x=0..1,n,shading=yellow,title="Midpoint Rule"):
display(array([Lplot,Rplot,Mplot]));
```

The `title` command lets you distinguish the graphs from each other. Notice that the left-hand sum is an underestimate, the right-hand sum is an overestimate, and the midpoint rule looks pretty good.

To evaluate the sums for one of the numerical approximations, we use the `leftsum`, `rightsum`, `middlesum`, `trapezoid`, or `simpson` commands. For example, typing

```
evalf(leftsum(exp(x^2),x=0..1,n));
```

gives the value of the left-hand sum for $f(x) = e^{x^2}$ over $0 \leq x \leq 1$ with 10 rectangles. The answer is 1.3812606013158460362. Similarly, typing

```
evalf(trapezoid(exp(x^2),x=0..1,n));
```

gives the value obtained by the trapezoid rule. The answer is 1.4671746927387982980. The other commands work in a similar fashion. If you want more rectangles (and thus a more accurate answer), replace n with a larger value. The `evalf` command (standing for "evaluate using floating-point arithmetic") is necessary to obtain a numerical answer.

Finally, we can numerically calculate a definite integral using the `int` command. This can also be used to do symbolic integration as well. Maple has its own built in numerical and symbolic integrators for computing integrals. For example, to find the numerical value of $\int_0^1 e^{x^2}\,dx$, type

```
evalf(int(exp(x^2),x=0..1));
```

The answer is 1.4626517459071816088. You can also choose the definite integral template from the "Expression" menu and fill in the desired values. To find the exact value of the integral (assuming this is possible), remove the `evalf` command from the previous line.

Exercises:

1. The following questions involve the function $f(x) = 4 - x^2$. Before executing the numerical calculations, type `Digits:=20:` which will output a greater number of significant digits in all of your computations. Recall that we define the error of an approximation to be

$$\text{Error} = |\text{Actual value} - \text{Approximation}|$$

   (a) Plot a graph of the left-hand sum, right-hand sum, and midpoint rule side by side for $f(x) = 4 - x^2$ over the interval $0 \le x \le 2$ with $n = 8$ subintervals. Turn in your graph. You may need to use the mouse to stretch the graph out to make a nicer figure.

   (b) Use the Fundamental Theorem of Calculus to compute the exact area under f over the interval $0 \le x \le 2$. You may check your answer with Maple.

   (c) Use Maple to calculate the left-hand and right-hand sums for $f(x) = 4 - x^2$ over the interval $0 \le x \le 2$ using $n = 2, 20, 200$, and 2000 rectangles. At each stage compute the error with the value obtained in part **(b)**. You can easily use Maple to compute the error. **Make a table** of your values and the corresponding errors.

   (d) Use Maple to calculate the midpoint and trapezoid rules for $f(x) = 4 - x^2$ over the interval $0 \le x \le 2$ using $n = 2, 20, 200$, and 2000 subintervals. At each stage compute the error (using Maple) with the value obtained in part **(b)**. Make a table of your values and the corresponding errors.

   (e) Use Maple to calculate Simpson's rule for $f(x) = 4 - x^2$ over the interval $0 \le x \le 2$ using $n = 2, 20, 200$, and 2000 subintervals. What do you notice? What is the error? Experiment by increasing the number of digits precision to 50 and 100.

   (f) For the left-hand, right-hand, midpoint, and trapezoid rules, discuss the improvement in each method as the number of subintervals increases. In other words, for each method, increasing n by a factor of 10 decreases the error by what factor? How many decimal places of accuracy are gained when n is increased by a factor of 10?

   (g) Compare the error of the midpoint rule and trapezoid rule. Which method is more accurate and by what factor? Your answer should help explain the definition of Simpson's rule.

2. The following questions involve the integral

$$\int_{-1}^{\pi} \sin(2x^2 + 1)\, dx\,.$$

   (a) Use the `int` and `evalf` commands to numerically approximate the integral using Maple. Note that this integral is impossible to do other than through numerical techniques, so this answer is the best we can hope for.

   (b) Use Maple to calculate the left-hand and right-hand sums for this integral using $n = 2, 20, 200$, and 2000 rectangles. Recall that π is entered as `Pi` in Maple. At each stage compute the error with the value obtained in part **(a)**. Make a table of your values and the corresponding errors.

   (c) Use Maple to calculate the midpoint and trapezoid rules for this integral using $n = 2, 20, 200$, and 2000 subintervals. At each stage compute the error with the value obtained in part **(a)**. Make a table of your values and the corresponding errors.

(d) Use Maple to calculate the integral with Simpson's rule using $n = 2, 20, 200,$ and 2000 subintervals. At each stage compute the error with the value obtained in part **(a)**. Make a table of your values and the corresponding errors.

(e) For all five numerical techniques, discuss the improvement in each method as the number of subintervals increases. In other words, for each method, increasing n by a factor of 10 decreases the error by what factor? For each method, how many decimal places of accuracy are gained when n is increased by a factor of 10? Which method is the most effective?

(f) Compare the error of the midpoint rule and trapezoid rule. Which method is more accurate and by what factor? Are there any differences with your answer to **1(g)**?

3. One fun and informative feature of the `student[Calculus1]` package is the ability to animate Riemann Sums. As the number of rectangles increases, the area under the curve is better approximated. The following command is an example of how to visualize this improvement for $f(x) = e^{-x^2}$ on the interval $1 \leq x \leq 2$ for 6 different partitions. The first partition you see should have 4 rectangles and should be a right-hand sum. Carefully type

```
ApproximateInt(exp(-x^2),1..2, output = animation, iterations = 6,
method = right, partition = 4, subpartition = all, refinement = halve);
```

To run the animation, click on the plot. On the toolbar at the top of the screen there should appear a DVD-like panel from which you can play the animation. The double arrow commands control the speed of the animation (you may wish to slow it down). The arrow with a vertical bar next to it will play one frame at a time. Notice how the area estimate improves with each iteration. Turn in the plot which is obtained after running the animation through once (this should be the graph with the best approximation).

4. The velocity of a spaceship in km/hr is given by

$$v(t) = 104.2\, t^3\, e^{-0.15\, t^2}$$

where $t \geq 0$ is measured in hours. Use Maple to calculate the distance the ship travels in the first 12 hours of its journey. How far does the ship travel in the next 12 hours? Explain how you obtained your answers.