MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 9.3: Slope Fields and Euler's Method

This section introduces some graphical and numerical methods for solving differential equations. The graphical technique involves sketching solution curves on the slope field defined by the ODE. This topic is the subject of **Computer Project #3**. In this worksheet we discuss a simple numerical technique called **Euler's Method**, which approximates a particular solution to a differential equation.

Euler's Method: Stepping Along the Slope Field

Recall that the slope field for the differential equation $\frac{dy}{dt} = f(t, y)$ is found by evaluating the function f at different points of the ty-plane and plotting tiny line segments with slope given by the value of f. Then, solutions to the differential equation must always be tangent to the slope field. Euler's Method, named after the great Swiss mathematician Leonhard Euler (pronounced "Oiler"), makes use of this concept to numerically approximate the solution beginning at the initial point (t_0, y_0). The basic idea is to step along the slope field, going a fixed length in the horizontal direction (Δt), and drawing line segments whose slope matches the one given by the slope field. In this fashion, a sequence of points (t_0, y_0), (t_1, y_1), $(t_2, y_2), \ldots$ is found that approximates the actual solution (see the figure below).

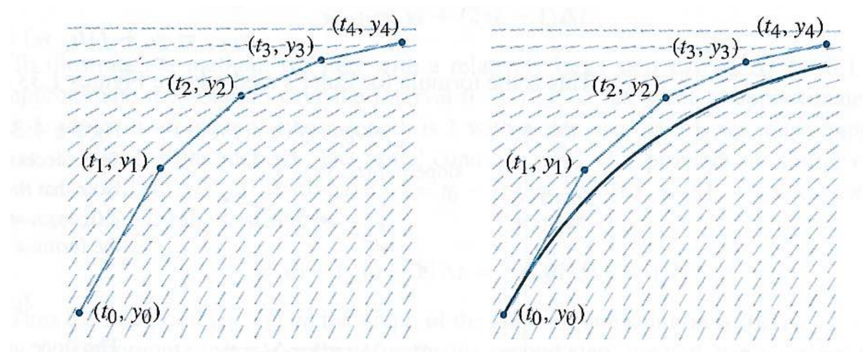

Figure 1.31
Stepping along the slope field.

Figure 1.32
The graph of a solution and its approximation obtained using Euler's method.

Figure 1: Four steps of Euler's method giving an approximate solution to the ODE beginning at the point (t_0, y_0). Figures from *Differential Equations, 4th ed.*, Blanchard, Devaney, and Hall, p. 53.

Suppose that we wish to approximate the solution to the ODE and initial condition

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \,.$$

To implement Euler's Method, we first define a **step size** Δt (called a **time step** in the textbook). This is the amount we increase the t-values by at each stage:

$$t_{k+1} = t_k + \Delta t \,.$$

Thus, $t_1 = t_0 + \Delta t$ and $t_2 = t_1 + \Delta t$, and so on. To find the next y-value from the previous one, we set the slope of the line segment between (t_k, y_k) and (t_{k+1}, y_{k+1}) to match $m_k = f(t_k, y_k)$. This gives

$$m_k = f(t_k, y_k) = \frac{y_{k+1} - y_k}{t_{k+1} - t_k} = \frac{y_{k+1} - y_k}{\Delta t} \quad \Longrightarrow \quad y_{k+1} = y_k + m_k \Delta t \,.$$

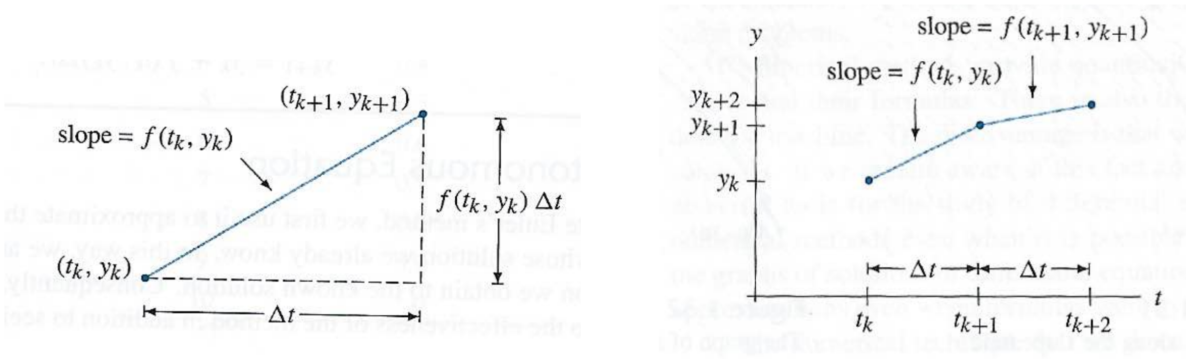

Figure 2: The figure on the left demonstrates one step of Euler's Method. The slope of the blue line segment is $f(t_k, y_k)$, which is found by evaluating the function on the right-hand side of the ODE at the point (t_k, y_k). The next segment of the solution has slope $f(t_{k+1}, y_{k+1})$, obtained by evaluating the function at the new point (t_{k+1}, y_{k+1}), as shown in the figure on the right. Figures from *Differential Equations, 4th ed.*, Blanchard, Devaney, and Hall, pp. 53–54.

Now the process repeats with the new slope $m_{k+1} = f(t_{k+1}, y_{k+1})$ (see Figure 2). Euler's Method is given by repeated iteration of the following two equations:

$$\boxed{\begin{aligned} t_{k+1} &= t_k + \Delta t \\ y_{k+1} &= y_k + f(t_k, y_k)\Delta t \end{aligned}}$$

Example 1: Use Euler's Method with a step size of $\Delta t = 0.2$ to approximate the value of $y(1)$, where $y(t)$ is the solution to

$$\frac{dy}{dt} \;=\; t + y, \quad y(0) = 1\,.$$

Answer: Let $f(t, y) = t + y$. We begin with the initial condition $y(0) = 1$ so that $t_0 = 0$ and $y_0 = 1$. The next t-value is $t_1 = t_0 + \Delta t = 0 + 0.2 = 0.2$, and the next y-value is

$$y_1 \;=\; y_0 + f(t_0, y_0)\Delta t \;=\; 1 + f(0, 1) \cdot 0.2 \;=\; 1 + 1 \cdot 0.2 \;=\; 1.2\,.$$

Now we repeat the process to find (t_2, y_2). The next t-value is $t_2 = t_1 + \Delta t = 0.2 + 0.2 = 0.4$, while the next y-value is

$$y_2 \;=\; y_1 + f(t_1, y_1)\Delta t \;=\; 1.2 + f(0.2, 1.2) \cdot 0.2 \;=\; 1.2 + 1.4 \cdot 0.2 \;=\; 1.48\,.$$

Since our goal is to approximate $y(1)$, we need three more steps to go from $t = 0.4$ to $t = 1$. The values obtained from Euler's Method are shown in the table below:

k	t_k	y_k	$m_k = f(t_k, y_k)$
0	0	1	1
1	0.2	1.2	1.4
2	0.4	1.48	1.88
3	0.6	1.856	2.456
4	0.8	2.3472	3.1472
5	1	2.97664	

Thus, our solution is $y(1) \approx 2.97664$.

Note the simple pattern in the column for t_k. The real effort lies in computing the values for y_k. For this example, it is possible to find the explicit formula for the solution to the differential equation. It is given by $y(t) = 2e^t - t - 1$. From this we see that the actual value of the solution is $y(1) = 2e - 2 \approx 3.4365637$. Thus the error in our Euler's Method approximation is

$$\text{Error} \;=\; |3.4365637 - 2.97664| \;=\; 0.459924\,.$$

Exercises:

1. Consider the ODE and initial condition $\dfrac{dy}{dt} = -2ty^2,\; y(1) = \dfrac{1}{2}\,.$

   (a) Use Euler's Method with a step size (time step) of $\Delta t = 0.25$ to estimate the value of $y(2)$.

   (b) Using the Separation of Variables technique, find an explicit formula for the particular solution to the ODE.

   (c) What is the error in your approximation to $y(2)$?

2. Use Euler's Method with a step size of $\Delta t = 0.1$ to estimate $y(0.5)$, where $y(t)$ is the solution to

$$\frac{dy}{dt} = 1 - 2y,\; y(0) = 2\,.$$

   Determine the error in your approximation by finding the particular solution to the ODE.