MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 9.2: Newton's Law of Cooling

This section explores a few applications arising from a simple linear differential equation. The applications we will discuss are Newton's Law of Cooling and annuities.

Newton's Law of Cooling

The rate at which the temperature of an object cools (or warms) is proportional to the difference between its temperature and that of its surrounding medium:

$$\frac{dy}{dt} = k(y - A).$$

Here $y(t)$ is the temperature of the object at time t, A is the ambient temperature of the surrounding medium (a constant), and k is the **cooling constant**.

In most problems, the value of k is not given; it has to be determined from the given information. If the object is cooling, then k is negative, while if the object is warming, then k is positive.

Example 1: Suppose a gold ring at a temperature of 400°F is immersed in a tank of water which has a constant temperature of 60°F. After 4 minutes, the temperature of the ring is down to 100°F.

(a) What is the temperature of the ring after 6 minutes?

(b) What is the temperature of the ring after 15 minutes?

(c) What is the temperature of the ring after a very, very long time?

Answer: Let $y(t)$ be the temperature of the ring at time t in minutes, and let $A = 60$ be the ambient temperature. We are given two pieces of information about the temperature of the ring: $y(0) = 400$ (the initial temperature of the ring) and $y(4) = 100$ (the temperature after 4 minutes).

Using Newton's Law of Cooling, we have

$$\frac{dy}{dt} = k(y - 60)\,.$$

Using the Separation of Variables technique, we have

$$\begin{aligned} \frac{dy}{y-60} = k\,dt \quad &\Longrightarrow \quad \ln|y-60| = kt + c \\ &\Longrightarrow \quad |y-60| = e^{kt+c} = ce^{kt} \\ &\Longrightarrow \quad y - 60 = ce^{kt} \\ &\Longrightarrow \quad y = 60 + ce^{kt}\,. \end{aligned}$$

Now we find the values of c and k. Since $y(0) = 400$, we have $400 = 60 + ce^0 = 60 + c$. Therefore $c = 340$. Then $y(4) = 100$ implies $100 = 60 + 340e^{4k}$, which gives in turn

$$\frac{40}{340} = e^{4k} \quad \Longrightarrow \quad k = \frac{1}{4}\ln\left(\frac{2}{17}\right) \approx -0.5350165.$$

Thus we have found the formula for the temperature, $y(t) = 60 + 340e^{-0.5350165\,t}$.

(a) $y(6) = 60 + 340e^{-0.5350165 \cdot 6} \approx 73.72°$ F.

(b) $y(15) = 60 + 340e^{-0.5350165 \cdot 15} \approx 60.11°$ F.

(c) Since $\lim_{t\to\infty} y(t) = 60$, over the long term, the temperature is settling down to 60°F. This makes sense because this is the temperature of the water in the tank.

Annuities: Investing While Withdrawing

An **annuity** is an investment in an account with regular withdrawals. The goal with an annuity is to earn enough money through compound interest so that it is possible to pay out some amount on a regular basis (e.g., a college's endowment). We will assume that money is compounded continuously at an annual rate of r, and that it is withdrawn continuously at a rate N. Our differential equation model is

$$\frac{dP}{dt} = rP - N = r\left(P - \frac{N}{r}\right).$$

The factored form of the equation is easier to solve.

Example 2: An annuity with a withdrawal rate of $N = \$3{,}000$ per year and interest rate 4% is funded with an initial deposit of $P_0 = \$40{,}000$.

(a) How much money is left in the account after 10 years?

(b) When will the annuity run out of funds?

Answer: We are given $N = 3000$, $r = 0.04$, and $P(0) = 40{,}000$. Then $N/r = 75{,}000$ and the ODE is $\frac{dP}{dt} = 0.04(P - 75{,}000)$. Using the Separation of Variables technique, we have

$$\begin{aligned}
\frac{dP}{P - 75{,}000} = 0.04\,dt \implies & \ln|P - 75{,}000| = 0.04t + c \\
\implies & |P - 75{,}000| = e^{0.04t+c} = ce^{0.04t} \\
\implies & P - 75{,}000 = ce^{0.04t} \\
\implies & P = 75{,}000 + ce^{0.04t}.
\end{aligned}$$

To find c, we use the initial condition $P(0) = 40{,}000$. This implies $40{,}000 = 75{,}000 + c$, so that $c = -35{,}000$. Therefore, the amount of money in the account is given by $P(t) = 75{,}000 - 35{,}000e^{0.04t}$.

(a) $P(10) = 75{,}000 - 35{,}000e^{0.04 \cdot 10} = \$22{,}786.14$.

(b) The annuity will run out of funds when $P(t) = 0$. Solving $75{,}000 - 35{,}000e^{0.04t} = 0$ gives

$$35{,}000e^{0.04t} = 75{,}000 \implies e^{0.04t} = \frac{15}{7} \implies t = \frac{1}{0.04}\ln\left(\frac{15}{7}\right) \approx 19.05 \text{ years.}$$

Exercises:

1. A cold metal bar at $-20°C$ is submerged in a pool maintained at a temperature of $50°C$. One minute later, the temperature of the bar is $10°C$. How long will it take for the bar to reach a temperature of $30°C$? What is the temperature of the bar after a very long time?

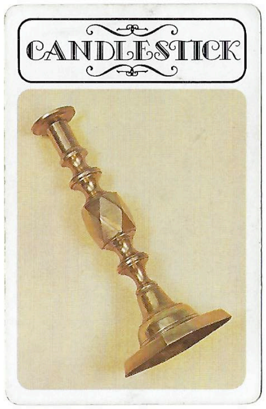
2. An annuity with a withdrawal rate of $N = \$6{,}000$ per year and interest rate 3% is funded with an initial deposit of $P_0 = \$60{,}000$.

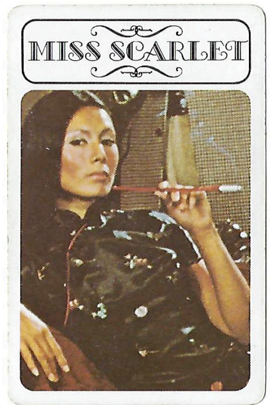
   (a) How much money is left in the account after 5 years?

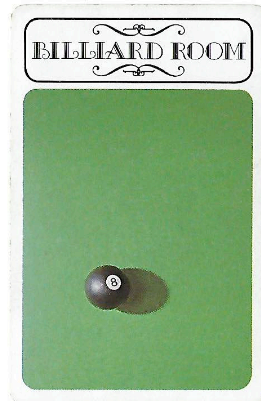
   (b) When will the annuity run out of funds?

Murder Mystery!

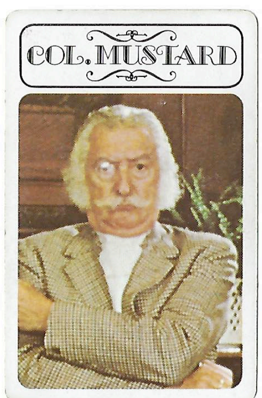
At 10:00 am you wander toward the pool table and find . . .

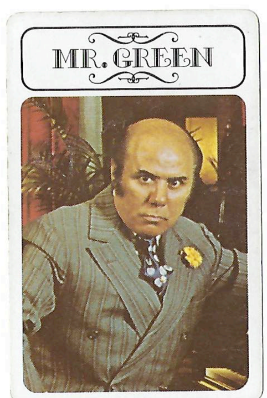






Stressed though you are by this terrible sight, you realize that you must catch the murderer. Fortunately, you have the presence of mind to measure the temperature of the body: 82.6°F. Then, you realize that you should round up the possible suspects:





- Colonel Mustard has an alibi from 3:00 pm – 5:00 pm and 9:30 pm – 5:00 am.
- Mr. Green has an alibi from 4:00 pm - 9:00 pm.
- Mrs. Peacock has an alibi from 8:00 pm – 1:00 am.

At 11:00 am you measure the temperature of the body again and find it is 81.7°F. The ambient temperature of the Billiard Room is kept at 72°F. Find the time of the murder to the nearest minute. Which suspect should you detain for questioning?