MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 8.1: Arc Length

In this section we will learn how to find the length of a curve, specifically, the length of the graph of a function. This is known as the **arc length**. The resulting integral is typically impossible to compute by hand so numerical integration techniques are often necessary to compute the arc length. For this worksheet (and on homework), we choose functions where the integrals are possible to do by hand or by using an integration table.

Arc Length Formula

To find the arc length of a differentiable function $f(x)$ from $x = a$ to $x = b$, we use

$$\boxed{\text{Arc Length} \;=\; \int_a^b \sqrt{1 + [f'(x)]^2}\, dx\,.} \tag{1}$$

The basic idea behind the formula is to draw small tangent vectors (line segments) at successive points along the graph, and then sum up the lengths of these vectors. Since the slope to $y = f(x)$ is given by the derivative dy/dx, a right triangle with length Δx for the base and height $f'(x) \cdot \Delta x$ will have a hypotenuse that approximates the curve. Using the Pythagorean Theorem, the length of the hypotenuse of this triangle is

$$\sqrt{(\Delta x)^2 + (f'(x)\Delta x)^2} \;=\; \sqrt{(\Delta x)^2(1 + [f'(x)]^2)} \;=\; \Delta x\sqrt{1 + [f'(x)]^2}\,.$$

Taking the limit as Δx approaches 0 and summing over the curve from $x = a$ to $x = b$ gives the integral in formula (1).

Example 1: Compute the arc length of the graph of $f(x) = 3x - 1$ from $x = 1$ to $x = 5$ using formula (1). Then check your answer using the distance formula.

Answer: We have $f'(x) = 3$ for any x, so formula (1) simply becomes

$$L \;=\; \int_1^5 \sqrt{1 + 3^2}\, dx \;=\; \int_1^5 \sqrt{10}\, dx \;=\; \sqrt{10}\int_1^5 1\, dx \;=\; \sqrt{10}\, x\Big|_1^5 \;=\; 4\sqrt{10}\,.$$

On the other hand, note that $f(x)$ is a linear function so its graph is just a line. Since $f(1) = 2$ and $f(5) = 14$, the arc length from $x = 1$ to $x = 5$ is the length of the line segment between the points $(1, 2)$ and $(5, 14)$. Using the distance formulas (essentially the Pythagorean Theorem), we have

$$L \;=\; \sqrt{(5-1)^2 + (14-2)^2} \;=\; \sqrt{16 + 144} \;=\; \sqrt{160} \;=\; 4\sqrt{10}\,,$$

which agrees with the answer obtained using formula (1).

Example 2: Compute the arc length of the graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from $x = 1$ to $x = e^2$.

Answer: The key to this problem is to write $1 + [f'(x)]^2$ as a perfect square. We compute $f'(x) = \frac{1}{4}x - x^{-1}$. Then,

$$1 + [f'(x)]^2 \;=\; 1 + \frac{1}{16}x^2 - \frac{1}{2} + x^{-2} \;=\; \frac{1}{16}x^2 + \frac{1}{2} + x^{-2} \;=\; \left(\frac{1}{4}x + x^{-1}\right)^2.$$

This implies that the arc length is

$$\int_1^{e^2} \sqrt{\left(\frac{1}{4}x + x^{-1}\right)^2}\, dx \;=\; \int_1^{e^2} \frac{1}{4}x + x^{-1}\, dx \;=\; \frac{1}{8}x^2 + \ln x\Big|_1^{e^2} \;=\; \frac{1}{8}e^4 + 2 - \left(\frac{1}{8} + 0\right) = \frac{1}{8}\left(e^4 + 15\right).$$

Exercises: For #1–3, compute the arc length of the graph of each function over the given interval.

1. $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

2. $f(x) = x^3 + \frac{1}{12}x^{-1}$ from $x = 1$ to $x = 2$.

3. $g(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ from $x = 0$ to $x = 3$.

4. Verify that the circumference of the unit circle is 2π by computing the arc length of the curve $y = \sqrt{1 - x^2}$ from $x = -1$ to $x = 1$.