MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 7.5: Partial Fractions

This section focuses on integrals involving rational functions $p(x)/q(x)$ where p and q are polynomials, and p has a lower degree than q. The main idea is to break the fraction up into pieces that are easily integrated. Splitting the fraction into smaller pieces is called a **partial fraction decomposition**.

Example 1: A Denominator with Distinct Linear Factors

Consider the integral $\displaystyle\int \frac{7x+13}{x^2+5x-14}\,dx$. Notice that the denominator factors as $(x+7)(x-2)$. To compute the partial fraction decomposition of the integrand, we seek constants A and B such that

$$\frac{7x+13}{x^2+5x-14} \;=\; \frac{A}{x+7}+\frac{B}{x-2}. \tag{1}$$

It turns out that whenever the denominator has distinct linear factors, it is always possible to find these special constants A and B (they are unique). Multiply both sides of equation (1) by the least common denominator $(x+7)(x-2)$. After cancelling, this gives

$$7x+13 \;=\; A(x-2)+B(x+7). \tag{2}$$

This equation needs to be satisfied **for any** x. Here is a useful tip: To find A and B, **plug in the roots** $x=2$ and $x=-7$. Plugging in $x=2$ into equation (2) gives $27 = A\cdot 0 + B\cdot 9$, which implies $B=3$. Plugging in $x=-7$ into equation (2) gives $-36 = A\cdot(-9)+B\cdot 0$, which implies $A=4$. To compute the integral, we break the fraction into two pieces, each of which can be integrated using simple u-substitutions:

$$\int \frac{7x+13}{x^2+5x-14}\,dx \;=\; \int \frac{4}{x+7}+\frac{3}{x-2}\,dx \;=\; 4\ln|x+7|+3\ln|x-2|+c\,.$$

In the final step, the two integrals are computed with u-substitutions $u=x+7$ and $u=x-2$, respectively. Note that $du=dx$ in both cases.

The technique above generalizes to other settings, provided we know how to compute the partial fraction decomposition of the integrand. Below are the relevant partial fraction decompositions we will make use of. The goal is to find the values of the unknown constants A, B, and C. Plugging in the roots for x is always a good idea. It is also useful to plug in easy x-values (e.g., $x=0$ or $x=1$.)

1. $\displaystyle\frac{p(x)}{(x-r_1)(x-r_2)} = \frac{A}{x-r_1}+\frac{B}{x-r_2}$ Two Distinct Linear Factors

2. $\displaystyle\frac{p(x)}{(x-r_1)(x-r_2)(x-r_3)} = \frac{A}{x-r_1}+\frac{B}{x-r_2}+\frac{C}{x-r_3}$ Three Distinct Linear Factors

3. $\displaystyle\frac{p(x)}{(x-r_1)(x-r_2)^2} = \frac{A}{x-r_1}+\frac{B}{x-r_2}+\frac{C}{(x-r_2)^2}$ Repeated Linear Factor

4. $\displaystyle\frac{p(x)}{(x-r_1)(x^2+a^2)} = \frac{A}{x-r_1}+\frac{Bx+C}{x^2+a^2}$ Irreducible Quadratic Factor

Example 2: A Denominator with an Irreducible Quadratic Factor

Consider the integral $\int \frac{6x^2-11x-1}{(x-3)(x^2+1)}\,dx$. Notice that x^2+1 cannot be factored (it is called **irreducible**). To compute the partial fraction decomposition of the integrand, we seek constants A, B, and C such that

$$\frac{6x^2-11x-1}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}\,. \tag{3}$$

Multiply both sides of equation (3) by the least common denominator $(x-3)(x^2+1)$. After cancelling, this gives

$$6x^2-11x-1 = A(x^2+1) + (Bx+C)(x-3). \tag{4}$$

First, we plug in the root $x=3$ into equation (4). This gives $20 = 10A$, which implies $A=2$. Since there are no more real roots, we next try plugging in simple x-values. Plugging in $x=0$ into equation (4) gives $-1 = A-3C$, which implies $C=1$ since we already know $A=2$. Plugging in $x=1$ into equation (4) gives $-6 = 2A-2(B+C)$, or $-8=-2B$ after substituting in $A=2$ and $C=1$. This gives $B=4$.

To compute the integral, we have

$$\begin{aligned}
\int \frac{6x^2-11x-1}{(x-3)(x^2+1)}\,dx &= \int \frac{2}{x-3} + \frac{4x+1}{x^2+1}\,dx \\
&= 2\ln|x-3| + \int \frac{4x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\
&= 2\ln|x-3| + 2\ln|x^2+1| + \tan^{-1}x + c\,.
\end{aligned}$$

In the final step, the middle integrals is computed with the u-substitution $u = x^2+1$. Here $du = 2x\,dx$ so we pull a 2 out of the integral.

Exercises: Evaluate each of the following integrals on a **separate** piece(s) of paper.

1. $\int \frac{3x+20}{x^2+4x}\,dx$

2. $\int \frac{2x^2+16}{x^3-4x}\,dx$

3. $\int \frac{x^2-16x}{(x+2)(x-1)^2}\,dx$

4. $\int \frac{10}{(x-1)(x^2+9)}\,dx$

5. $\int \frac{1}{(x-1)^2(x-2)^2}\,dx$