MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 7.6: Strategies for Integration

Solutions

Exercises: Evaluate each of the following integrals by choosing an appropriate technique(s) of integration.

1. $\int x\sqrt{x^2+1}\,dx$

   Answer: This is easiest to compute using a u-substitution. Let $u = x^2+1$. Then $du = 2x\,dx$ and we have

$$\int x\sqrt{x^2+1}\,dx \;=\; \frac{1}{2}\int 2x\sqrt{x^2+1}\,dx \;=\; \frac{1}{2}\int u^{1/2}\,du \;=\; \frac{1}{2}\cdot\frac{2}{3}u^{3/2}+c \;=\; \frac{1}{3}\left(x^2+1\right)^{3/2}+c\,.$$

2. $\int x^3\sqrt{x^2+1}\,dx$

   Answer: This can also be computed using a u-substitution, which is easier than doing trig sub. Let $u = x^2+1$. Then $du = 2x\,dx$ and $x^2 = u-1$. If we split the x^3 term into $x\cdot x^2$, we can evaluate the integral using the power rule. We find

$$\begin{aligned}
\int x^3\sqrt{x^2+1}\,dx &= \frac{1}{2}\int 2x\cdot x^2\sqrt{x^2+1}\,dx \\
&= \frac{1}{2}\int (u-1)u^{1/2}\,du \\
&= \frac{1}{2}\int u^{3/2}-u^{1/2}\,du \\
&= \frac{1}{2}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\,du \\
&= \frac{1}{5}(x^2+1)^{5/2}-\frac{1}{3}(x^2+1)^{3/2}+c\,.
\end{aligned}$$

3. $\int t^2e^{3t}\,dt$

   Answer: This integral can be evaluated using integration by parts twice. First, let $u = t^2$ and $dv = e^{3t}\,dt$. Then $du = 2t\,dt$ and $v = \frac{1}{3}e^{3t}$. We find

$$\int t^2e^{3t}\,dt \;=\; t^2\cdot\frac{1}{3}e^{3t}-\int\frac{1}{3}e^{3t}\cdot 2t\,dt \;=\; \frac{1}{3}t^2e^{3t}-\frac{2}{3}\int te^{3t}\,dt\,.$$

   The remaining integral can be done using integration by parts again. Let $u = t$ and $dv = e^{3t}\,dt$.

Then $du = dt$ and $v = \frac{1}{3}e^{3t}$. We compute

$$\begin{aligned}
\int t^2 e^{3t}\, dt &= \frac{1}{3}t^2 e^{3t} - \frac{2}{3}\int te^{3t}\, dt \\
&= \frac{1}{3}t^2 e^{3t} - \frac{2}{3}\left(t \cdot \frac{1}{3}e^{3t} - \int \frac{1}{3}e^{3t}\, dt\right) \\
&= \frac{1}{3}t^2 e^{3t} - \frac{2}{3}\left(\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + c\right) \\
&= \frac{1}{3}t^2 e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{27}e^{3t} + c \\
&= \frac{1}{27}e^{3t}\left(9t^2 - 6t + 2\right) + c\,.
\end{aligned}$$

4. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^4\theta\, d\theta$

Answer: To evaluate this integral, first notice that the integrand is even. By symmetry, we have $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^4\theta\, d\theta = 2\int_0^{\pi/4} \sin^4\theta\, d\theta$. Next we apply the formula $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$. Being careful to FOIL the integrand, we find

$$\int_{-\pi/4}^{\pi/4} \sin^4\theta\, d\theta \;=\; 2\int_0^{\pi/4}\left(\frac{1}{2}(1-\cos(2\theta))\right)^2 d\theta \;=\; \frac{1}{2}\int_0^{\pi/4} 1 - 2\cos(2\theta) + \cos^2(2\theta)\, d\theta\,.$$

Next we use the formula $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$, except we replace θ by 2θ. We find

$$\begin{aligned}
\frac{1}{2}\int_0^{\pi/4} 1 - 2\cos(2\theta) + \cos^2(2\theta)\, d\theta &= \frac{1}{2}\int_0^{\pi/4} 1 - 2\cos(2\theta) + \frac{1}{2}(1+\cos(4\theta))\, d\theta \\
&= \frac{1}{2}\int_0^{\pi/4} \frac{3}{2} - 2\cos(2\theta) + \frac{1}{2}\cos(4\theta)\, d\theta \\
&= \frac{1}{2}\left(\frac{3}{2}\theta - \sin(2\theta) + \frac{1}{8}\sin(4\theta)\Big|_0^{\pi/4}\right) \\
&= \frac{1}{2}\left(\frac{3}{2}\cdot\frac{\pi}{4} - \sin(\pi/2) + \frac{1}{8}\sin(\pi) - 0\right) \\
&= \frac{3\pi}{16} - \frac{1}{2}\,.
\end{aligned}$$

5. $\displaystyle\int \frac{x^2}{(9-x^2)^{3/2}}\, dx$ *Hint:* After making a trig substitution, use the identity $\tan^2\theta + 1 = \sec^2\theta$ to compute the integral.

Answer: This problem can be computed using the trig substitution $x = 3\sin\theta$. Then $dx =$

$3\cos\theta\, d\theta$ and $9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta$. We find

$$\begin{aligned}
\int \frac{x^2}{(9-x^2)^{3/2}}\,dx &= \int \frac{9\sin^2\theta}{(9\cos^2\theta)^{3/2}} \cdot 3\cos\theta\, d\theta \\
&= \int \frac{27\sin^2\theta\cos\theta}{27\cos^3\theta}\, d\theta \\
&= \int \tan^2\theta\, d\theta \\
&= \int \sec^2\theta - 1\, d\theta \\
&= \tan\theta - \theta + c \\
&= \frac{x}{\sqrt{9-x^2}} - \sin^{-1}\left(\frac{x}{3}\right) + c\,.
\end{aligned}$$

The final step comes from using right-triangle trig and drawing a right triangle with legs x and $\sqrt{9-x^2}$ and hypotenuse 3.

6. $\displaystyle\int \frac{11x+6}{(x^2+4)(x-3)}\,dx$

Answer: This problem uses partial fractions. We write

$$\frac{11x+6}{(x^2+4)(x-3)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}$$

and search for the constants A, B, and C. Multiplying through on both sides by $(x-3)(x^2+4)$, we find

$$11x + 6 = A(x^2+4) + (Bx+C)(x-3)\,.$$

Plugging in $x = 3$ gives $39 = 13A$ so that $A = 3$. Plugging in $x = 0$ yields $6 = 4A - 3C$ so that $C = 2$. Finally, plugging in $x = 1$ gives $17 = 5A - 2(B+C) = 11 - 2B$ so that $B = -3$. Thus we have

$$\begin{aligned}
\int \frac{11x+6}{(x^2+4)(x-3)}\,dx &= \int \frac{3}{x-3} + \frac{-3x+2}{x^2+4}\,dx \\
&= 3\ln|x-3| + \int \frac{-3x}{x^2+4}\,dx + \int \frac{2}{x^2+4}\,dx \\
&= 3\ln|x-3| - \frac{3}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{2}{x^2+4}\,dx \\
&= 3\ln|x-3| - \frac{3}{2}\ln|x^2+4| + \tan^{-1}\left(\frac{x}{2}\right) + c\,,
\end{aligned}$$

where the last two integrals are done by u-sub ($u = x^2+4$) and trig sub ($x = 2\tan\theta$), respectively.

7. $\displaystyle\int \sin^{-1}(\sqrt{x})\,dx$ *Hint:* Start with the u-substitution $u = \sqrt{x}$. Then just go for it.

Answer: Letting $u = \sqrt{x}$, we have $du = \frac{1}{2}x^{-1/2}\,dx$ and thus $dx = 2\sqrt{x}\,du = 2u\,du$. Thus,

$$\int \sin^{-1}(\sqrt{x})\,dx = 2\int \sin^{-1}(u)\cdot u\,du\,.$$

Now use integration by parts with $t = \sin^{-1}(u)$ and $dv = u\,du$. This gives $dt = \frac{1}{\sqrt{1-u^2}}\,du$ and $v = u^2/2$. Thus,

$$2\int \sin^{-1}(u)\cdot u\,du \;=\; 2\sin^{-1}(u)\cdot\frac{u^2}{2} - 2\int \frac{u^2}{2\sqrt{1-u^2}}\,du \;=\; u^2\sin^{-1}u - \int\frac{u^2}{\sqrt{1-u^2}}\,du\,.$$

The remaining integral can be computed using the trig sub $u = \sin\theta$. Then $du = \cos\theta\,d\theta$ and $\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. We find

$$\begin{aligned}
\int \frac{u^2}{\sqrt{1-u^2}}\,du &= \int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\,d\theta \\
&= \int \sin^2\theta\,d\theta \\
&= \int \frac{1}{2}\left(1-\cos(2\theta)\right)\,d\theta \\
&= \frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + c \\
&= \frac{1}{2}\left(\theta - \sin\theta\cos\theta\right) + c \\
&= \frac{1}{2}\left(\sin^{-1}u - u\cdot\sqrt{1-u^2}\right) + c\,.
\end{aligned}$$

Finally, putting everything together and returning to the original variable x, we have

$$\begin{aligned}
\int \sin^{-1}(\sqrt{x}\,)\,dx &= 2\int \sin^{-1}(u)\cdot u\,du \\
&= u^2\sin^{-1}u - \int\frac{u^2}{\sqrt{1-u^2}}\,du \\
&= u^2\sin^{-1}u - \left(\frac{1}{2}\left(\sin^{-1}u - u\cdot\sqrt{1-u^2}\right)\right) + c \\
&= u^2\sin^{-1}u - \frac{1}{2}\sin^{-1}u + \frac{1}{2}u\cdot\sqrt{1-u^2} + c \\
&= x\sin^{-1}(\sqrt{x}) - \frac{1}{2}\sin^{-1}(\sqrt{x}) + \frac{1}{2}\sqrt{x}\cdot\sqrt{1-x} + c\,.
\end{aligned}$$