MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 5.7: The Substitution Method

Recall the chain rule:

$$\frac{d}{dx}\left[f(u(x))\right] \;=\; f'(u(x)) \cdot u'(x).$$

If we take the integral of both sides, we find

$$\int f'(u)\,du \;=\; \int f'(u(x)) \cdot u'(x)\,dx \;=\; \int \frac{d}{dx}\left[f(u(x))\right]\,dx \;=\; f(u(x)).$$

This suggests a technique for finding an antiderivative: determine the "inside" function $u(x)$ and make a substitution, called a *u-sub* for short, that turns the integrand into a simpler integral in the variable u. The technique of **u-substitution** essentially uses the chain-rule backwards.

Example 0.1 *Evaluate* $\int 6x(3x^2+7)^8\,dx$ *using the substitution* $u = 3x^2 + 7$.

Since $u = 3x^2 + 7$, $\frac{du}{dx} = 6x$ *or* $du = 6x\,dx$. *Making this substitution, the integral transforms into*

$$\int u^8\,du \;=\; \frac{1}{9}u^9 + c \;=\; \frac{1}{9}(3x^2+7)^9 + c.$$

Note that we return to the variable x at the end of the problem. We can easily check our answer by using the chain rule:

$$\frac{d}{dx}\left[\frac{1}{9}(3x^2+7)^9 + c\right] \;=\; (3x^2+7)^8 \cdot 6x \;=\; 6x(3x^2+7)^8,$$

as desired. Here is another example.

Example 0.2 *Evaluate* $\int xe^{-x^2}\,dx$ *using the substitution* $u = -x^2$.

Here we have $u = -x^2$ *so that* $\frac{du}{dx} = -2x$ *or* $du = -2x\,dx$. *However, we have an* $x\,dx$ *term in the integrand; we are missing a factor of* -2. *We use the simple trick of multiplying and dividing the integrand by* -2 *to make it look the way we want, remembering that constants pull out of integrals.*

$$\int xe^{-x^2}\,dx \;=\; \int -\frac{1}{2}\cdot -2xe^{-x^2}\,dx \;=\; -\frac{1}{2}\int -2xe^{-x^2}\,dx \;=\; -\frac{1}{2}\int e^u\,du \;=\; -\frac{1}{2}e^u + c \;=\; -\frac{1}{2}e^{-x^2} + c.$$

Alternatively, we could have solved $\frac{du}{dx} = -2x$ for dx, obtaining $dx = \frac{du}{-2x}$ and then made the substitution. Either way, we obtain the same integral in the variable u. The key to the u-sub technique of integration is to find the correct substitution u and then transform the integral into an easier one that *only involves the variable u.*

Our third example explains how to use u-substitution with a definite integral.

Example 0.3 *Evaluate* $\int_0^1 x^2(1+2x^3)^5\,dx$ *using the substitution* $u = 1 + 2x^3$.

Since $u = 1 + 2x^3$, *we have* $du = 6x^2\,dx$ *so we need to multiply the integrand by 6 and pull out the constant* $\frac{1}{6}$. *But we also need to change the limits of integration. If* $x = 0$, *then* $u = 1 + 2(0)^3 = 1$ *and if* $x = 1$, *then* $u = 1 + 2(1)^3 = 3$. *Thus, our definite integral becomes*

$$\int_0^1 x^2(1+2x^3)^5\,dx \;=\; \int_0^1 \frac{1}{6}\cdot 6x^2(1+2x^3)^5\,dx \;=\; \frac{1}{6}\int_1^3 u^5\,du \;=\; \frac{1}{36}u^6\Big|_1^3 \;=\; \frac{1}{36}(3^6-1) \;=\; \frac{182}{9}.$$

Exercises:

1. Evaluate $\int 3x^3 \sqrt{6x^4+1}\, dx$ using the substitution $u = 6x^4 + 1$.

2. Evaluate $\int \frac{4t}{t^2+1}\, dt$ using the substitution $u = t^2 + 1$.

3. Evaluate $\int \tan\theta \, d\theta$ using the substitution $u = \cos\theta$. Why won't the substitution $u = \sin\theta$ work?

4. Evaluate $\int (x-2)\sqrt{x+1}\, dx$ using the substitution $u = x + 1$.

5. Evaluate $\int \frac{\cos x + 4}{(\sin x + 4x)^3}\, dx$.

6. Evaluate $\int_1^e \frac{\ln x}{x}\, dx$.

7. Evaluate $\int_{-1}^{2} \sqrt{5x+6}\, dx$

8. Evaluate $\int_0^{\pi/4} \sin^3(2\theta)\cos(2\theta)\, d\theta$.

9. Evaluate $\int_{-5}^{5} \frac{x^5 - 3x^3 + 7x}{x^6+4}\, dx$.