MATH 136-02, 136-03 Calculus 2, Fall 2018

Section 5.3: The Indefinite Integral

This section focuses on using the differentiation process in the opposite direction: instead of finding the derivative of a function f, we find a function F whose derivative equals f. The function F is known as an antiderivative of f ("anti" = "opposite").

Definition 0.1 *If $F'(x) = f(x)$ over an interval (a, b), then F is called an* **antiderivative** *of f.*

Example 0.2 *$F(x) = x^2$ is an antiderivative of $f(x) = 2x$ because $\frac{d}{dx}\left(x^2\right) = 2x$.*

Example 0.3 *$G(x) = \sin x$ is an antiderivative of $g(x) = \cos x$ because $\frac{d}{dx}(\sin x) = \cos x$.*

Some Important Notes:

- To check whether a function F is an antiderivative of f, simply take the derivative of F and see if it equals f. In other words, check that $F'(x) = f(x)$.
- Notice that $F(x) = x^2 + 5$ and $F(x) = x^2 - \pi$ are also antiderivatives of $f(x) = 2x$ because the derivative of a constant is zero. In general, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + c$ for **any** constant c. There are an **infinite** number of antiderivatives for any one given function. From a graphical perspective, $F(x) + c$ is a vertical shift of the graph of $F(x)$. Since a vertical translation does not alter the slope of the graph, all functions of the form $F(x) + c$ have the same derivative.

Exercise 1: Find an antiderivative $F(x)$ for $f(x) = \sin x$ satisfying $F(0) = 5$.

The Indefinite Integral

The process of finding an antiderivative is called **integration**. We use the integral sign $\int$, without any limits of integration, to indicate a general antiderivative. In other words,

$$\int f(x)\, dx \;=\; F(x) + c \quad \text{means that} \quad F'(x) = f(x).$$

The expression $\int f(x)\, dx$ is called an **indefinite integral**. Note that there are *no* limits of integration. The reason why we use the same symbol for an antiderivative as we did for the signed area under the curve (the definite integral) will become clear when we study the Fundamental Theorem of Calculus.

Exercise 2: Explain why $\int 3x^2\, dx = x^3 + c$ and $\int \cos(4\theta)\, d\theta = \frac{1}{4}\sin(4\theta) + c$.

Key Integration Formulas: $c, k \in \mathbb{R}$ are arbitrary constants

1. $\int 0\, dx = c$, where c is an arbitrary constant

2. $\int k\, dx = kx + c$

3. **Power Rule**: $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c, \quad$ where $n \neq -1$

4. $\int \frac{1}{x}\, dx = \ln|x| + c$

5. $\int e^x\, dx = e^x + c$ and more generally, $\int e^{kx}\, dx = \frac{1}{k} e^{kx} + c, \quad (k \neq 0)$

6. $\int a^x\, dx = \frac{a^x}{\ln a} + c$ for any real number $a > 0$

7. $\int \sin x\, dx = -\cos x + c$ and more generally, $\int \sin(kx)\, dx = -\frac{1}{k}\cos(kx) + c, \quad (k \neq 0)$

8. $\int \cos x\, dx = \sin x + c$ and more generally, $\int \cos(kx)\, dx = \frac{1}{k}\sin(kx) + c, \quad (k \neq 0)$

9. $\int \sec^2 x\, dx = \tan x + c$

10. $\int \csc^2 x\, dx = -\cot x + c$

11. $\int \sec x \tan x\, dx = \sec x + c$

12. $\int \csc x \cot x\, dx = -\csc x + c$

13. **Linearity: (i)** $\int k f(x)\, dx = k \int f(x)\, dx$ (constants pull out)

    (ii) $\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$ (integral of a sum is the sum of the integrals)

Exercise 3: Notice the absolute value signs in Formula 4. Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0\,. \end{cases}$$

Using this definition and the chain rule, check that Formula 4 is correct for both $x > 0$ and $x < 0$.

Exercise 4: Use the chain rule to verify Formulas 5 and 7:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \qquad \text{and} \qquad \int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + c, \quad (k \neq 0)$$

Exercise 5: Find each indefinite integral.

(a) $\displaystyle\int 4\cos(2x) - 5\sin(3x)\,dx$

(b) $\displaystyle\int \frac{3}{x^2} + 7e^{-2x}\,dx$

Exercise 6: Find an antiderivative $F(x)$ of $f(x) = \dfrac{1}{\sqrt{x}} + 5x^3$ satisfying $F(1) = 3$.

Note: For homework purposes, this is the same thing as solving the initial value problem

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} + 5x^3, \quad y(1) = 3$$

(see Exercises 47–62 on p. 287 of the textbook).

Exercise 7: Suppose that $f''(x) = -5\sin x + 72x$ and $f(0) = 6$ and $f'(0) = 1$. Find $f(x)$.

Exercise 8: A particle moves on a line with acceleration given by $a(t) = 4t + 6$ m/s^2. If the initial velocity is $v(0) = -7$ m/s and the initial position is $s(0) = 25$ m, find the position function $s(t)$.