MATH 136-02, 136-03 Calculus 2, Fall 2018

The FUNdamental Theorem of Calculus

Sections 5.4 and 5.5

This worksheet focuses on the most important theorem in calculus. In fact, the Fundamental Theorem of Calculus (FTC) is arguably one of the most important theorems in all of mathematics. In essence, it states that differentiation and integration are inverse processes. There are two parts to the FTC, the second of which is the most difficult to understand.

The Fundamental Theorem of Calculus (FTC)
Suppose that f is a continuous function over a closed interval $a \leq x \leq b$.

Part 1: $\displaystyle\int_a^b f(x)dx \;=\; F(b) - F(a)$, where F is any antiderivative of f.

Part 2: If $A(x) = \displaystyle\int_a^x f(t)\,dt$ is the area function giving the area under f from a to x, then $A(x)$ is differentiable and its derivative is just $f(x)$. In other words, $A'(x) = f(x)$ or

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) \;=\; f(x). \tag{1}$$

Some Important Notes Concerning the FTC:

(i) The first part of the FTC will be used over and over again throughout the course. It states that to find the signed area under f from a to b, we need only find an antiderivative of f, evaluate it at the endpoints a and b, and then subtract. There is something fundamentally surprising going on here: somehow, the area only depends on the values of the antiderivative at the endpoints, not in between. How is this possible? It is a deep fact and one that has a useful generalization into higher dimensions via an important theorem known as *Stokes Theorem*.

(ii) Since F is an antiderivative of F' (by definition), another way to write part 1 of the FTC is

$$\int_a^b F'(t)\,dt \;=\; F(b) - F(a). \tag{2}$$

Notice the integral sign and derivative sign canceling out here. Equation (2) basically says that integrating a rate of change gives the **net** (or total) change. We have already seen this in terms of Physics with the relationship between position $s(t)$ and velocity $v(t)$:

$$\int_a^b s'(t)\,dt \;=\; \int_a^b v(t)\,dt \;=\; s(b) - s(a).$$

In words, the integral of velocity over $a \leq t \leq b$ gives the total change in position (sometimes called the net displacement). An example from Economics would be something like

$$\int_a^b C'(x)\,dx \;=\; C(b) - C(a),$$

which states that the integral of the **marginal cost** gives the total change in cost when increasing the number of items produced from a to b units.

(iii) In part 2 of the FTC, the variable is x and it is the upper limit of integration. The lower limit a is an arbitrary constant. As x varies, the area under the curve varies, which is why $A(x)$ is really a function. The amazing aspect of part 2 is that this function has a derivative equal to the function we are finding the area under, namely f.

(iv) A simple way to remember part 2, and in particular equation (1), is that the derivative and integral sign cancel out. In other words, differentiation and integration are inverse processes — doing one operation and then the other gets you back to where you started. But be careful, the variable x must be a **limit of integration!** Check out one of my favorite final exam questions below:

Exercise 0: Calculate

$$\frac{d}{dx}\left(\int_0^7 \sin(e^{t^2})\,dt\right)$$

The answer is NOT $\sin(e^{x^2})$.

(v) It is particularly important to understand the difference between $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$. The limits of integration, or lack thereof, are critical. The $\int f(x)\,dx$ is called an **indefinite integral** and represents the general antiderivative of $f(x)$. Here you can think of the integral sign as telling you to find the general antiderivative. For example, we have

$$\int x^2\,dx = \frac{1}{3}x^3 + c \quad \text{and} \quad \int \cos x\,dx = \sin x + c\,.$$

The quantity $\int_a^b f(x)\,dx$ is called the **definite integral**, and represents the signed area under f from a to b. The definite integral is a **number**!

Exercises: (problems #1–3 focus on part 1 of the FTC, while the remaining problems use part 2)

1. Use part 1 of the FTC to compute $\displaystyle\int_{-1}^{2} 2x+1\,dx$.

   Check that your answer agrees with the one you obtained for Example 0.3 on the worksheet for Section 5.2 (The Definite Integral).

2. Use part 1 of the FTC to compute $\displaystyle\int_0^{\pi} \sin\theta\,d\theta$ and $\displaystyle\int_0^{2\pi} \sin\theta\,d\theta$.

   Interpret your answers graphically.

3. Evaluate each integral using part 1 of the FTC:

(a) $\displaystyle\int_0^1 5e^{3t} + \sqrt{t}\, dt$

(b) $\displaystyle\int_1^3 \frac{2}{x} - \frac{5}{x^2}\, dx$

(c) $\displaystyle\int_{\pi/3}^{\pi/2} \sin(3\theta) + 4\cos(2\theta)\, d\theta$

(d) $\displaystyle\int_0^4 |x^2 - 4x + 3|\, dx$ *Hint:* Draw a graph of the integrand and then break the integral up into three pieces without any absolute value signs.

4. If $F(x) = \int_{-3}^{x} \sqrt[3]{\cos^2(3t) + 5}\, dt$, find $F'(x)$.

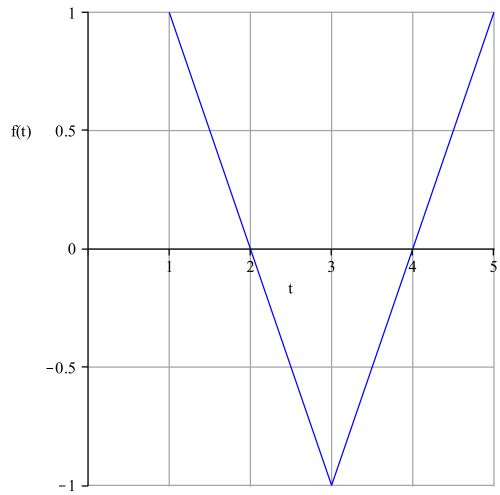
5. Find $\dfrac{d}{dx}\left(\int_{8}^{x} e^{t^2}\, dt\right)$, and then find $\dfrac{d}{dx}\left(\int_{x}^{8} e^{t^2}\, dt\right)$.

6. Find $\dfrac{d}{dx}\left(\int_{4}^{\sqrt{x}} \sin(t^2)\, dt\right)$. *Hint:* Use the chain rule where the "inside" function is $\sqrt{x}$.

7. If $G(x) = \int_{x^3}^{5} \sqrt{t^5 + 5}\, dt$, find $G'(x)$.

8. If $H(x) = \int_{x^3}^{e^{2x}} \sqrt{t^5 + 5}\, dt$, find $H'(x)$. *Hint:* Break the integral into two integrals. Then differentiate.

9. Suppose that $G(x) = \int_1^x f(t)\,dt$, where the graph of f is shown below.



(a) Find $G(1), G(2), G(3), G(4)$ and $G(5)$.

(b) Find $G'(2), G'(3)$ and $G'(5)$.

(c) Where is $G(x)$ concave up? Does $G''(3)$ exist? Explain.

(d) Sketch the graph of G over the interval $1 \leq x \leq 5$.